Complexity of the Minimum Cost Homomorphism Problem for Semicomplete Digraphs with Possible Loops

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김은정

Definitions : Graph Homomorphism

- Homomorphism
  For digraphs D and H, a mapping $f : V(D) \to V(H)$ is a *homomorphism* of D to H if $uv \in A(D)$ implies $f(u)f(v) \in A(H)$.

- H-coloring
  A homomorphism f of G to H is also called an H*-coloring* of G, and $f(x)$ is called the *color* of the vertex x in G

- Homomorphism Problem
  Let H be a fixed directed or undirected graph. The *homomorphism problem*, HOM(H), for H asks whether a directed or undirected input graph G admits a homomorphism to H.

- List Homomorphism Problem
  The *list homomorphism problem*, ListHOM(H), for H asks whether a directed or undirected input graph G with lists (sets) $L_u \subseteq V(H)$, admits a homomorphism f to H in which $f(u) \in L_u$ for each $u \in V(G)$.

Definitions : Graph Homomorphism

- Min Cost Homomorphism Problem
  Suppose G and H are directed (or undirected) graphs, and $c_i(u)$, $u \in V(G)$, $i \in V(H)$ are nonnegative *costs*. The *cost of a homomorphism* f of G to H is $\sum_{u \in V(G)} c_{f(u)}(u)$. If H is fixed, the *minimum cost homomorphism problem*, MinHOM(H), for H is the following optimization problem. Given an input graph G, together with costs $c_i(u)$, $u \in V(G)$, $i \in V(H)$, find a minimum cost homomorphism of G to H, or state that none exists.

- Relationship between HOM(H), ListHOM(H) and MinHOM(H).
  HOM(H) $\rightarrow$ (polytime reducible) ListHOM(H) $\rightarrow$ (polytime reducible) MinHOM(H).

More Definitions

- Tournament

  A loopless digraph D is a *tournament* if there is exactly one arc between every pair of vertices.

- Semicomplete Digraph

  A loopless digraph D is a *semicomplete digraph* if there is at least one arc between every pair of vertices.

- With Possible Loops

  We will consider *semicomplete digraphs with possible loops (w.p.l.)*, i.e., digraphs obtained from semicomplete digraphs by appending some number of loops (possibly zero loops).

- Reflexive, Loopless

  If a directed (undirected) graph G has no loops, we call G *loopless*. If a directed (undirected) graph G has a loop at every vertex, we call G *reflexive*.

Constraint Satisfaction Problem

- Constraint Satisfaction Problem
  Given a vocabulary (finite number of relation symbols, each of finite arity), and two structures G and H over the vocabulary, with ground set $V(G)$ and $V(H)$, and interpretations of all the relation symbols, as relations over the set of the stated arities.
- More Definition
  - Let A be a finite set. An $n-$*ary relation* on A is a set of $n-$tuples of elements from A.
  - R_A denotes the set of all finitary relations on A.
  - *A constraint language* is a subset of R_A, and my be finite or infinite.

Constraint Satisfaction Problem

- CSP; More Formally

  The Constraint Satisfaction Problem over a constraint language $H \subseteq R_A$, denoted CSP(H), is defined to be the decision problem with instance $\mathcal{P} = (V; A; \mathcal{C})$, where

  – V is a finite set of *variable*

  – A is a *set of values*, and

  – $\mathcal{C}$ is a set of *constraints*, in which each constraint $C \in \mathcal{C}$ is a pair $\langle s, R \rangle$ with s a list of variables of length m_C, called the *constraint scope*, and R an m_C-ary relation on A, belonging to H, called the *constraint relation*.

  The question is whether there exists a *solution* to $\mathcal{P}$, that is, a mapping $\varphi : V \mapsto A$ such that, for each constraint in $\mathcal{C}$, the image of the constrain scope is a member of the constraint relation.

Background

- If $P \neq NP$, as usually believed to be true, there is a problem in NP class of "intermediate" difficulty, i.e. which is neither polynomial time solvable nor NP-complete.
- Recent researches try to answer the following question: "What is the most general subclass of NP that we can define that may not contain such in-between problem?"
- One prominent approach in this direction is the study of (restricted) Constraint Satisfaction Problems, trying to prove the following conjecture.

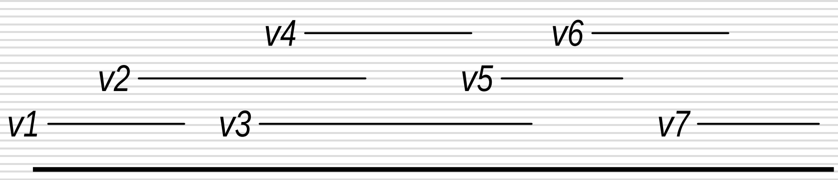
  Conjecture 1. *Any arbitrary restricted CSP is either polynomial time solvable or NP-complete.*
- Graph Homomorphism problems are a special form of CSP. To obtain a dichotomy classification of graph homomorphism problem is of interest from this point of view.

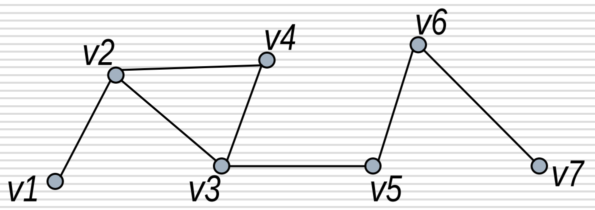
Previous Dichotomy Results

- Especially, homomorphism problem for directed graphs are considered to be difficult, and it remains a challenging open problem to obtain a dichotomy for HOM(H), ListHOM(H) and MinHOM(H) when H a directed graph.

Definitions : Intersection Graphs

- Intersection Graph
  - Given a family F={S1,S2,...Sn} of sets, define a graph G with V(G)=F in which Si and Sj are adjacent if and only if they are intersecting. G is called an intersection graph of F.
- Interval Graph
  - A graph isomorphic to the intersection graph of a family of intervals on the real line.
- Proper Interval Graph
  - If the intervals can be chosen to be inclusion-free.





Notation

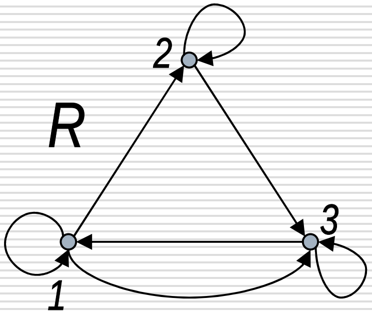
- Domination
  - For a digraph D, if *xy* is an arc of D, we say *x dominates y* and *y is dominated* by *x*.
  - If *x y* is an arc of D and *yx* is not an arc of D, we say x *strictly dominates y*.
  - Same notation for set of vertices.
- Symmetric, Assymetric arcs
  - If *x y* is an *assymetric* arc if *x* strictly dominates *y*.
  - If both *x y* and *yx* are arcs of D, *x y* is a *symmetric* arc.
- Symmetric part of a digraph
  - H^{sym} denotes the *symmetric subdigraph* of H, i.e. a digraph with the same vertex set with H and all symmetric arcs of H.

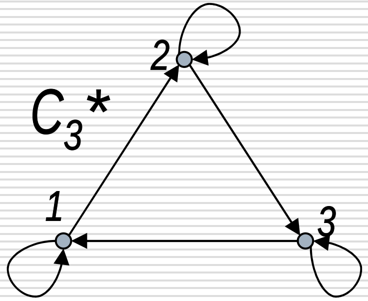
Contents

- Dichotomy : H is a reflexive semicomplete digraph
  - Proof of NP-hard part
  - Proof of Polynomial Solvable part
- Dichotomy : H is a semicomplete digraph w.p.l
  - Proof of NP-hard part
  - Proof of Polynomial Solvable part

H : Reflexive Semicomplete Digraph

- R : V(R)={1,2,3},
  A(R)={12,23,31,13,11,22,33}
- C_3* : a reflexive directed cycle
  on three vertices

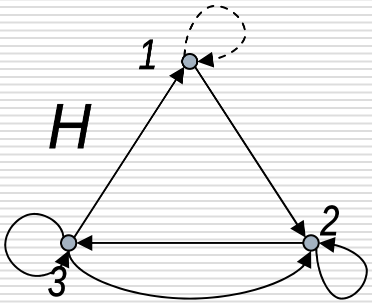




- Main Dichotomy Result
  - Let H be a reflexive semicomplete digraph.
  - MinHOM(H) polynomial time solvable.
    - when H does not contain either R or C_3* as an induced subdigraph, and H^{sym} is a proper interval graph
  - MinHOM(H) NP-hard
    - Otherwise.

Proof of NP-hard case

- Basic observation
  - Let H' be an induced subdigraph of a digraph H. If MinHOM(H') is NP-hard, then MinHOM(H) is also NP-hard.

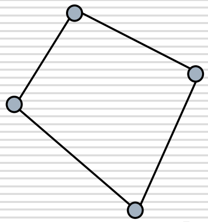


- Some lemmas
  - Let H be a digraph as depicted.

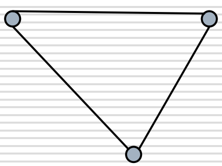
    Then MinHOM(H) is NP-hard.
    - Proof : Polynomial transfrom from MAX-SIZE INDEPENDENT SET problem. For details, see the paper.
  - Let a digraph H be obtained from C_k, k >2, by adding at least one loop. Then MinHOM(H) is NP-hard.

Proof of NP-hard case

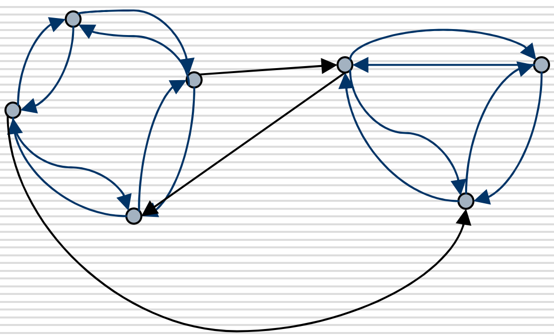
- One more lemma
  - Let H be a reflexive graph. If H is a proper interval graph (possibly with more than one component), then MinHOM(H) is polynomial time solvable. Otherwise, it is NP-hard.



Not proper interval



Proper interval



- Proof of NP-hard part
  - H - reflexive semicomplete, one of the followings
    - Contains R
    - Contains C_3*
    - At least one component of H^{sym} is not proper interval

Min-Max Ordering

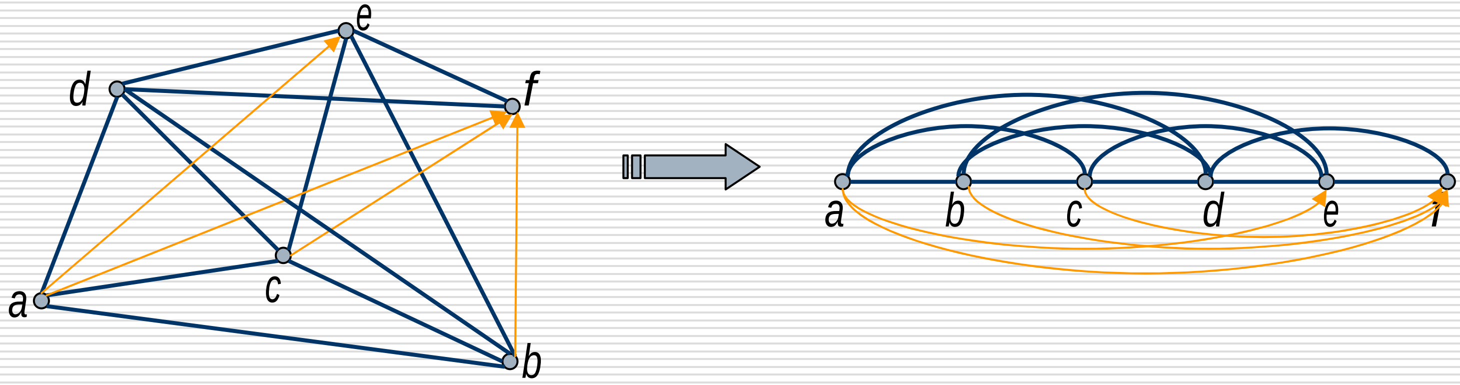
- Min-Max ordering
  - Let H be a digraph.
  - Let 1,2,...,n be an ordering of V(H).
  - $e = ir$ and $f = js$ be two arcs in H.
  - The pair min{i,j}min{r,s} and max{i,j}max{r,s} : the minimum and the maximum of e & f. (not necessarily arcs)
  - An ordering 1,2,...,n is a *Min-Max ordering* of V(H) if both minimum and maximum of every pair of arcs in H are again in A(H).
- Min-Max theorem
  - If a digraph H has a Min-Max ordering, MinHOM(H) is polynomial time solvable.

Proof Outline

- Basic assumption & Recall that
  - Assume that a reflexive semicomplete digraph H does not contain either R or C_3*, and H^{sym} is a proper interval graph.
  - For all other cases, we proved that MinHOM(H) is NP-hard
- Overall plan
  - We will show that H has a Min-Max ordering.
  - Arrange V(H) so that they have the Min-Max property.
  - Design an ordering of V(H) based on a specific ordering of a component of V(H^{sym}).
- Some lemma
  - A reflexive graph H is a proper interval graph if and only if its vertices can be ordered 1,2,...,n so that $i < j < k$ and $ik \in E(H)$ imply that $ij \in E(H)$ and $jk \in E(H)$

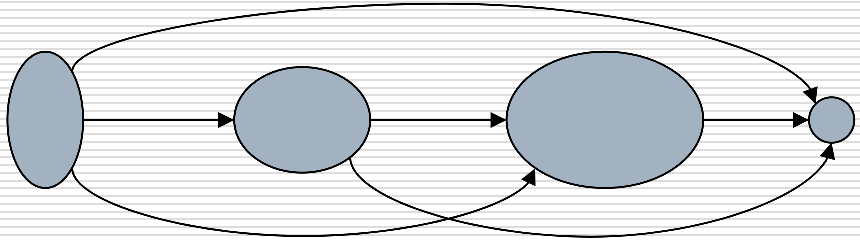
Ordering the vertices

- Ordering within a component of $V(H^{sym})$
  - Assumption : H is R-free + H^{sym} is a *connected* proper interval graph.
  - Then the vertices of H can be ordered 1,2,…,n so that for every pair of vertices *i* and *j* with $i < j$, *i* dominates *j* .
- Proof
  - Order the vertices 1,2,…,n so that $i < j < k$ and $ij \in E(H)$ imply that $jk \in E(H)$ and $ik \in E(H)$.



Ordering the vertices

- Ordering of $V(H^{sym})$
  - Assumption : H is R/ C_3*-free + H^{sym} is a proper interval graph (possibly more than one component).
  - Then the vertices of H can be ordered 1,2,…,n so that for every pair of vertices *i* and *j* with $i < j$, *i* dominates *j* .
- Proof
  - Acyclic ordering of the components of $V(H^{sym})$.
  - Within each component, the specified ordering above.



• Each, induced subdigraph by the vertices of a component of H^{sym}

• Between each pair, complete *strict domination*.

Proof of Polynomial case

- Lemma
  - The ordering of vertices by the previously described way is a Min-Max ordering of H.
- Theorem
  - Let H be a reflexive semicomplete digraph. If H does not contain either R or C_3* and H^{sym} is a proper interval graph, then MinHOM(H) is polynomial time solvable.
- Corollary
  - Suppose P is not equal to NP and let H be a reflexive semicomplete digraph. Then MinHOM(H) is polynomial time solvable if and only if H has a Min-Max ordering.

H : Semicomplete Digraph w.p.l

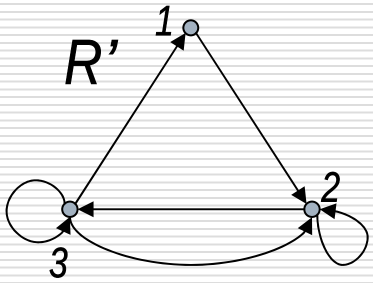
- Semicomplete digraph with possible loops

- W : V(W)={1,2},

  A(W)={12,21,22}

- R' : V(W)={1,2,3},

  A(W)={12,23,32,31,22,33}



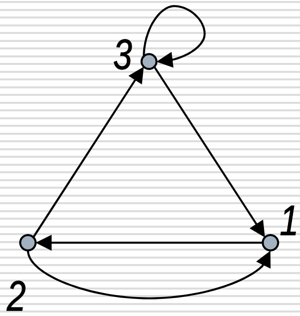
- Given a semicomplete digraph H w.p.l
  - let L(H)=L and I(H)=I be the maximal induced subdigraphs of H which are reflexive and loopless, respectively.

Main Dichotomy Result

- Main Dichotomy Result
  - Let H be a semicomplete digraph w.p.l.
  - MinHOM(H) polynomial time solvable.
    (a) when H is C_k for k=2 or 3.
    (b) when H
      - L does not contain either R or C_3* as an induced subdigraph
      - L^{sym} is a proper interval graph
      - I is a transitive tournament
      - H does not contain W, R', C_3 with at least one loop
  - MinHOM(H) NP-hard
    - otherwise.

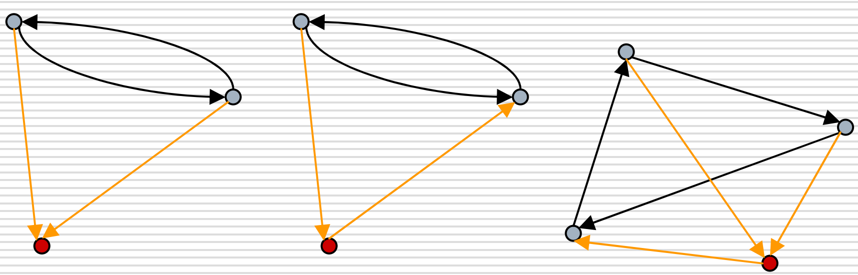
Proof of NP-hard case

- Some lemmas
  - MinHOM(W) is NP-hard.
  - MinHOM(R') is NP-hard.
  - Let H be a digraph as depicted.
    Then MinHOM(H) is NP-hard.
  - Let H' be a digraph obtained from C_k =12…k1, k >1, by adding an extra vertex k +1 dominated by at least two vertices of the cycle and let H" be the digraph obtained from H' by adding the loop at vertex k +1. Let H be H' or its converse or H" or its converse. Then MinHOM(H) is NP-hard.
  - Let H be a (loopless) semicomplete digraph with at least two directed cycles. Then the problem of checking whether a digraph D has an H-coloring is NP-complete.

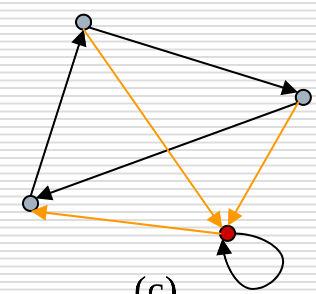


Proof of NP-hard case

- Let H be a semicomplete digraph w.p.l. If one of the following conditions holds, then MinHOM(H) is NP-hard
  - (a) I contains a cycle and I is not C_k for k=2 or 3.
  - (b) L contains either R or C_3* as an induced subdigraph, or L^{sym} is not a proper interval graph.
  - (c) I= C_k for k=2 or 3, and L is nonempty.
  - (d) H contains W, R' or C_3 with at least one loop.



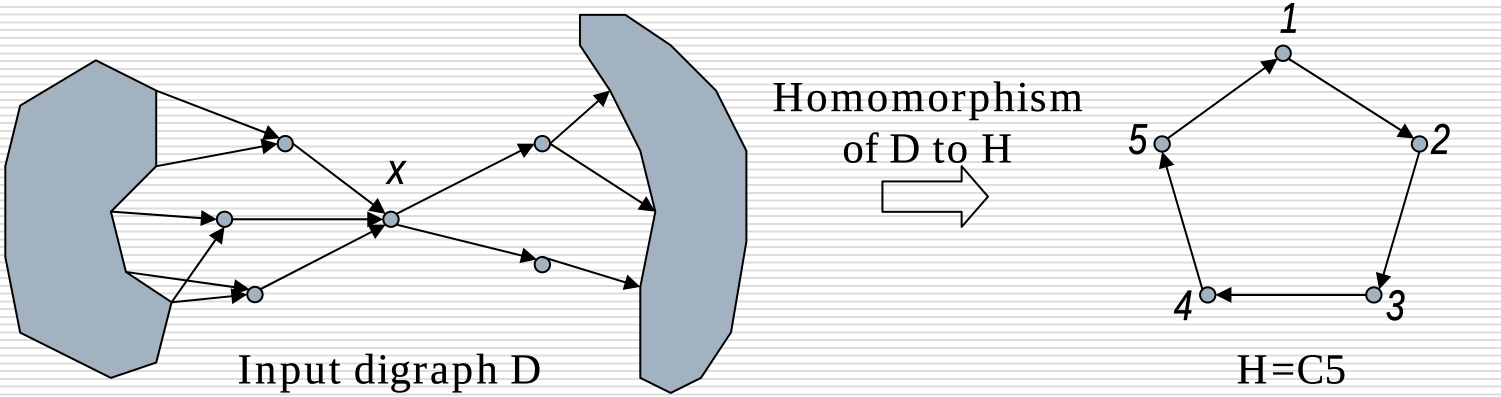
(a)



(c)

Proof of Polynomial case

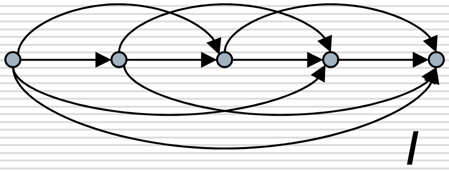
- When H=C_k, k >1 (condition (a) in the main theorem)
  - Polynomial time algorithm for solving MinHOM(H) exists.

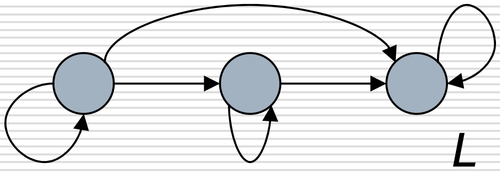


  - Choose a vertex x of D, assign it color 1.
  - For any y vertex with color i, we assign all the in-neighbors of y color i -1, all the out-neighbors of y color i +1.
  - Cyclicly permute the coloring of V(D), and calculate the cost for each assignment.

Ordering the vertices

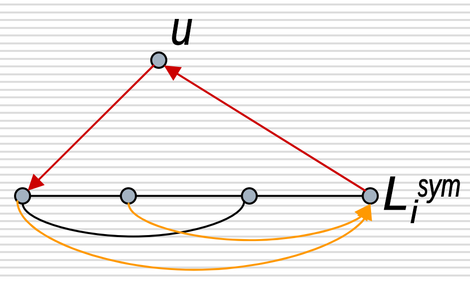
- Ordering of V(H)
  - Then the vertices of H can be ordered 1,2,…,n so that for every pair of vertices *i* and *j* with $i < j$, *i* dominates *j* .
- Proof outline
  - Acyclic ordering of V(I) : $u_1,\ldots,u_p$
  - Ordering of V(L) according to specified way above.
  - Especially, acyclic ordering of the components L_i^{sym}'s of L^{sym}
  - We will prove the statement by showing that the subdigraph induced by $V(L_i^{sym})$ of L^{sym} can be 'inserted' into an appropriate position among $u_1,\ldots,u_p$ without creating a cycle.



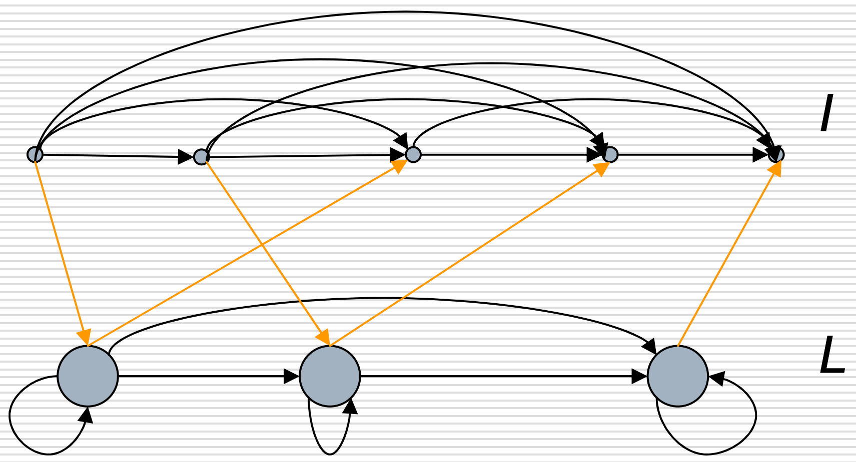


Proof Outline

- Construction of the ordering
  - First, we show that given a vertex *u* of I and a component L_i^{sym}, either u strictly dominates or strictly dominated by $V(L_i^{sym})$.



  - Each component L_i^{sym} can be inserted among V(I) so that...
  - The insertion of all $V(L_i^{sym})$'s does not change their relative order.



Proof of Polynomial case

- Lemma
  - The ordering of vertices by the previously described way is a Min-Max ordering of H.
- Theorem
  - Let H be a semicomplete digraph w.p.l. If H satisfies condition (b) in the main dichotomy theorem, then MinHOM(H) is polynomial time solvable.
- Equivalently..
  - Let H be a semicomplete digraph w.p.l. If H is a composition digraph s.t $H=TT_p[S_1,\ldots,S_k]$ where S_i is either a single vertex without a loop or a R-free reflexive semicomplete digraph whose symmetric part is a connected proper interval graph, then MinHOM(H) is NP-hard.