ELSEVIER Discrete Applied Mathematics 72 (1997) 219–241

DISCRETE APPLIED MATHEMATICS

Lifting cover inequalities for the precedence-constrained knapsack problem

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Received 8 August 1994; revised 7 September 1995

Abstract

This paper considers the polyhedral structure of the precedence-constrained knapsack problem, which is a knapsack problem with precedence constraints imposed on the set of variables. The problem itself appears in many applications. Moreover, since the precedence constraints appear in many important integer programming problems, the polyhedral results can be used to develop cutting-plane algorithms for more general applications. In this paper, we propose a modification of the cover inequality, which explicitly considers the precedence constraints. A combinatorial, easily implementable lifting procedure of the modified cover inequality is given. The procedure can generate strong cuts very easily. We also propose an additional lifting procedure, which is a generalization of the lifting procedure for cover inequalities. Some properties of the lifted inequality are analyzed and the problem of finding an optimal order of lifting is also addressed.

Keywords: Lifting procedure; Cutting-plane algorithm; Knapsack problem; Cover inequality

1. Introduction

This paper considers the polyhedral structure of the precedence-constrained knapsack problem (PK), which is a knapsack problem with precedence constraints on the set of variables. Formally, the problem can be stated as follows. Suppose a set $N = \{1, \ldots, n\}$ and a partial order $\leqslant$ are given. The partial order $\leqslant$ enforces that if $i \in S \subseteq N$ and $i' \leqslant i$, then $i' \in S$. For each $i \in N$, $a_i \in Z_+$ and $w_i \in Z$ are given and a positive integer b is given. The problem is to find a partially ordered set $S \subseteq N$ such that $\sum_{i \in S} a_i \leqslant b$ and maximizes $\sum_{i \in S} w_i$. By introducing a variable x_i for each $i \in N$,

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SSDI 0166-218X(95)00113-1

the problem can be formulated as the following integer programming problem.

$$\begin{aligned}
\text{(PK)} \quad \max \quad & \sum_{i \in N} w_i x_i \\
\text{s.t.} \quad & \sum_{i \in N} a_i x_i \leqslant b, \\
& x_i \geqslant x_j, \quad \text{if } i \leqslant j \\
& x_i \in \{0, 1\}, \quad \text{for all } i \in N.
\end{aligned}$$

The problem itself appears in many applications, for instance, see [5, 9, 4]. Moreover, since precedence constraints are a common characteristic of a number of important integer programming problems, the polyhedral results on (PK) can be used in devising a cutting-plane algorithm for more general applications, as is the case in [3], where they have used the polyhedral results on the knapsack problem. Note that (PK) is a knapsack problem with logical constraints. Chajakis and Guignard [2] considered a similar problem and presented exact algorithms for the problem.

Ibarra and Kim [6] considered the problem and have shown that the problem is NP-complete in the strong sense. Boyd [1] considered the polyhedral structure of the problem, which can be viewed as a generalization of the results on the knapsack polytope. He proposed several different methods to strengthen the cover inequality for some special cases. All the methods he proposed are special cases of the lifting procedure presented in this paper. Johnson et al. [7] considered a special case of (PK), which is obtained by linearizing the quadratic knapsack problem. This paper generalizes both results based on the lifting procedure to strengthen the well-known cover inequality for the knapsack polytope [10], which explicitly uses the precedence constraints. Some of the results given in this paper can be found in [12]. We include them here for completeness.

The paper is organized as follows. In Section 2, some notations and definitions used frequently are introduced. Section 3 presents preliminary results on the polyhedral structure of (PK). Some modifications of the cover inequality are shown there. Section 4 presents a lifting procedure and a proof of its validity. Section 5 considers the properties of the cover inequality lifted by the procedure given in the previous section. Section 6 addresses the problem of finding an optimal order of lifting. Section 7 considers the linear programming relaxation of (PK) and shows that all fractional solutions can be cut off by the cover inequality lifted by the proposed procedure. In Section 8, a lifting heuristic which further strengthens the lifted cover inequality is presented. Finally, Section 9 presents concluding remarks.

2. Notations and definitions

In this section, we introduce some notations and definitions which will be used frequently. Table 1 summarizes the notations used in the paper.

Table 1
Summary of notation

For a given graph $H = (V(H), A(H))$, a set $C \subseteq V(H)$ and an element $i \in V(H)$
$R(i) = \{j \in V(H) \mid (j,i) \in A(H)\}$ (set of predecessors)
$R(C) = \bigcup_{i \in C} R(i)$
$T(C) = R(C) \cup C$
$S(i) = \{j \in V(H) \mid (i,j) \in A(H)\}$ (set of successors)
$S(C) = \bigcup_{i \in C} S(i)$
$S_C(i) = \{j \in C \mid (i,j) \in A(H)\}$
$U_C(i) = \{i\} \cup S_C(i)$
$d_H^+(i) = \lvert S(i) \rvert$
$R_2(C) = \{i \in R(C) \mid \lvert S_C(i) \rvert \geqslant 2\}$ and $R_1(C) = R(C) \backslash R_2(C)$
$L(H)$: the set of leaves of H
$M(H)$: the set of co-leaves of H
$H(C) = (C \cup R_2(C), A_H(C))$, where $A_H(C)$ is the subset of arcs in $A(H)$ which have both ends in $C \cup R_2(C)$.

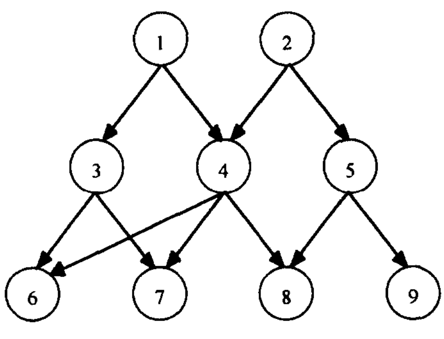


Fig. 1. Example of precedence graph.

Let $N = \{1, \ldots, n\}$ be the set of indices. For a given instance of (PK), let us define the *precedence graph* $G = (V(G), A(G))$ as follows.

$$V(G) = N, (i,j) \in A(G) \text{ if and only if } i \preccurlyeq j.$$

Without loss of generality, we can assume G has no directed cycles. In the following, all the graphs considered have no directed cycles. Fig. 1 is an example of a precedence graph where some arcs whose existence is clear from the transitive property of the partial order are not shown.

For a given graph $H = (V(H), A(H))$, a set $C \subseteq V(H)$ and an element $i \in V(H)$, let us define

$$R(i) = \{j \in V(H) \mid (j, i) \in A(H)\} \text{ (set of } \textit{predecessors}\text{)},$$

$$R(C) = \bigcup_{i \in C} R(i),$$

$$T(C) = R(C) \cup C,$$

$$S(i) = \{j \in V(H) \mid (i, j) \in A(H)\} \text{ (set of } \textit{successors}\text{)},$$

$$S(C) = \bigcup_{i \in C} S(i),$$

$$S_C(i) = \{j \in C \mid (i, j) \in A(H)\},$$

$$U_C(i) = \{i\} \cup S_C(i).$$

When the above notations are used, the underlying graph will be clear from the context. For each node $i \in V(H)$, let $d_H^+(i) = |S(i)|$. A node $i \in V(H)$ is called a leaf of H if $d_H^+(i) = 0$ and $L(H)$ is the set of leaves of H. A node $i \in V(H)$ is called a co-leaf of H if $S(i) \subseteq L(H)$. $M(H)$ is the set of co-leaves of H. For $C \subseteq V(H)$, let us partition $R(C)$ into two disjoint subsets $R_1(C)$ and $R_2(C)$, where $R_2(C) = \{i \in R(C) \mid |S_C(i)| \geqslant 2\}$ and $R_1(C) = R(C) \backslash R_2(C)$. Also define a graph $H(C) = (C \cup R_2(C), A_H(C))$, where $A_H(C)$ is the subset of arcs in $A(H)$ which have both ends in $C \cup R_2(C)$. For example, for the graph shown in Fig. 1, if we let $C = \{6, 7, 8, 9\}$,

$$R(C) = R_2(C) = \{1, 2, 3, 4, 5\}, \quad S(\{3, 4\}) = \{6, 7, 8\}, \quad L(G) = \{6, 7, 8, 9\},$$

$$M(G) = \{3, 4, 5\}.$$

Two elements i, j in N are called *incomparable* if neither $i \leqslant j$ nor $j \leqslant i$ holds. A subset $C \subseteq N$ is called *incomparable* if any two distinct elements of C are incomparable. Now let us introduce an operation on a graph H. For $i \in M(H)$, the *aggregated graph* H_1 *with respect to* i is obtained by replacing the set of nodes $U_{L(H)}(i)$ with a single new node and joining all the arcs which have one end in $U_{L(H)}(i)$ to the node. If there occur parallel arcs, replace them with a single arc. For notational convenience, we use the following convention. In the graph H, we assume the node i represents the singleton $\{i\}$, and if we obtain the aggregated graph H_1 with respect to $i, i \in M(H)$, the newly introduced node I represents the set $U_{L(H)}(i)$. The convention can be applied recursively.

For any set $C \subseteq N$ and a vector $a \in R^n$, $a(C)$ denotes $\sum_{i \in C} a_i$ and x^C denotes the characteristic vector of C, that is, $x_i^C = 1$, for all $i \in C$ and 0, otherwise. Throughout the paper, it is assumed that the reader is familiar with basic polyhedral theory (for example, see [10]). Let P be the set of feasible solutions to (PK). For any incomparable set $C \subseteq N$, let us define

$$P(C) = \operatorname{conv}(\{x^D \in P \mid D \subseteq T(C)\}).$$

Note that $P(C)$ is the convex hull of the feasible solutions to (PK) restricted to those variables in $T(C)$.
In addition, let $Q(C) = \text{conv}(\{x^D \in P \mid D \subseteq T(C) \text{ and } i \in D \text{ for all } i \in R_2(C)\})$, that is, $Q(C)$ is obtained from $P(C)$ by setting $x_i = 1$ for all $i \in R_2(C)$.

Throughout the paper, without loss of generality, we assume $a(T(i)) \leqslant b$ holds for all $i \in N$. Hence we can assume, for any incomparable set $C \subseteq N$, $P(C)$ is full-dimensional.

3. The induced cover inequalities

Let an instance of (PK) and the associated precedence graph G be given. From the well-known results on the polyhedral structure of the knapsack polytope, a subset $C \subseteq N$ is called a *cover* if $a(C) > b$ [10]. The associated inequality $x(C) \leqslant |C| - 1$ is called a *cover inequality*. A cover is called a *minimal cover* if no proper subset of it is a cover. When there exist precedence constraints on the set of variables, the following modification of the cover is more useful.

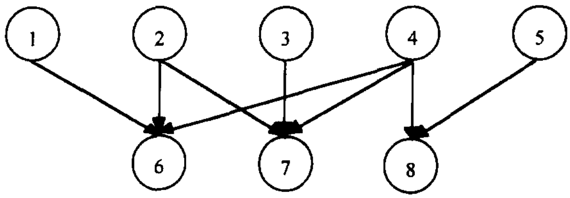
Definition 3.1. $C \subseteq N$ is called an induced cover (IC) if it is incomparable and $a(T(C)) > b$.

For any IC $C, x(C) \leqslant |C| - 1$ is a valid inequality and called an IC inequality. Note that if we have found a cover C, then by removing those nodes which have successors in C, we can obtain an IC. Moreover, the resulting IC inequality is at least as strong as the original cover inequality associated with the cover C. Hence, the additional restriction of incomparability causes no essential loss in the strength of the cover inequality.

Definition 3.2. An IC C is called a minimal induced cover (MIC) if $a(T(C \setminus \{i\})) \leqslant b$ for all $i \in C$.

For example, in Fig. 1, $C = \{6, 7, 8, 9\}$ is an MIC. The MIC inequality can be defined similarly. From any IC, we can obtain an MIC by successively removing those elements violating the condition in Definition 3.2. From any minimal cover, we can obtain an MIC by a similar method. The concept of an MIC differs from that of minimal cover introduced by Boyd [1]. He also restricted the cover to the case where all of the elements in it are incomparable, but he termed a cover minimal if $a(T(C) \setminus \{i\}) \leqslant b$, for all $i \in C$ [1].

However, there are some important applications [5, 7, 9] where we cannot construct a minimal cover in Boyd's sense from a given cover. Fig. 2 illustrates such a case, where the example is taken from an application that appeared in [5]. Note that the variables corresponding to C do not show up in the knapsack constraint. In



$V(G) = \{1, \ldots, 8\}$, Knapsack constraint : $\sum_{i=1}^{5} x_i \le 4$, $C = \{6,7,8\}$

Fig. 2. Example of an MIC.

general, a minimal cover C in Boyd's sense is also an MIC, but the converse does not hold (for example, see also Fig. 2).

The following result can be found in [1].

Theorem 3.1. *For a given $C \subseteq N$, suppose $T(C)$ is a minimal cover (or C is a minimal cover in Boyd's sense). The inequality $x(C) \leqslant |C| - 1$ is a facet of $P(C)$ if and only if $T(i) \cap T(j) = \emptyset$ for all $i, j \in C$, $i \neq j$.*

Hence for an MIC inequality related to a minimal cover $T(C)$ (or that related to a minimal cover in Boyd's sense) to define a facet of $P(C)$, $R_2(C)$ should be empty. In the following, we set the complicating variables in $R_2(C)$ to 1 (hence, $Q(C)$ results) and propose a lifting procedure which makes the resulting inequality define a facet of $P(C)$ whenever $T(C)$ is a minimal cover (or C is a minimal cover in Boyd's sense).

The following proposition can be proved easily.

Proposition 3.2. *For an MIC C, $a(R_2(C)) \leqslant b$.*

Hence if C is an MIC, $Q(C)$ is nonempty. Moreover, it can be easily shown that $Q(C)$ is full-dimensional, that is, of dimension $|C| + |R_1(C)|$. For an MIC C, the following result holds.

Proposition 3.3. *For an MIC C, the inequality $x(C) \leqslant |C| - 1$ is a facet of $Q(C)$ if and only if $a(T(C\setminus\{i\})) + a(T(j)) \leqslant b$ for all $j \in R(i) \cap R_1(C)$ and $i \in C$.*

Proof. The "if" part can be proved easily by showing $|C| + |R_1(C)|$ linearly independent points satisfying the MIC inequality at equality. The "only if" part can be proved as follows. Suppose the condition fails for some $i \in C$, $j \in R(i) \cap R_1(C)$. Then the following inequality is a valid inequality:

$$\sum_{k \in C\setminus\{i\}} x_k + x_j \leqslant |C| - 1.$$

Since $x_j \geqslant x_i$ holds, the inequality is stronger than the original MIC inequality. $\square$

Note that if $T(C)$ is a minimal cover (or C is a minimal cover in Boyd's sense), the condition in Proposition 3.3 trivially holds. If the condition holds for a given MIC, the associated MIC inequality can be lifted to define a facet of $P(C)$ by the procedure given in the next section. However, there are some important cases where an MIC inequality which does not satisfy the condition can also be lifted to define a facet of $P(C)$. Note that the MIC shown in Fig. 2 does not satisfy the above condition. However, in Section 5, we show that the associated lifted MIC inequality defines a facet of $P(C)$.

In the remainder of this section, we briefly mention the separation problem for IC inequalities. In general, the problem is NP-hard since it involves the separation problem for cover inequalities for the knapsack problem [10]. Therefore, we resort to some heuristic procedures to find violated IC inequalities. A possible heuristic procedure is as follows. First, find a cover by using a separation heuristic for cover inequalities for the knapsack problem. Then, make it an IC by the method mentioned above. When more restrictions are present on the structure of the precedence graph, it is possible to devise more efficient problem specific separation procedures. For example, in [11], some separation heuristics for the case where $G(C)$ is bipartite, were proposed.

4. Lifting procedure

Let us assume an MIC C is given. In the previous section, we showed that if C is an MIC, $Q(C)$ is not empty and the MIC inequality $x(C) \leqslant |C| - 1$ is valid for the polytope. In this section, we present a lifting procedure of the inequality on the set of variables in $R_2(C)$, which are set to 1 in $Q(C)$. For details of the general lifting procedure, see [10].

Let us denote the lifted cover inequality as follows:

$$\sum_{i \in C} x_i - \sum_{i \in R_2(C)} \gamma_i x_i \leqslant |C| - 1 - \sum_{i \in R_2(C)} \gamma_i.$$

The following is the procedure to determine γ_i for all $i \in R_2(C)$.

Procedure A

(*Initialization*) $H_0 = G(C)$, $k = 1$.

(*Step k*) If H_{k-1} has no arc, stop.

Otherwise, choose arbitrary $k \in M(H_{k-1})$.

Set $\gamma_k = d^+_{H_{k-1}}(k) - 1$.

Construct the aggregated graph from H_{k-1} with respect to k and set the resulting graph H_k.

$k = k + 1$. Go to step k.

At any step k in the procedure, we choose a variable x_k to be lifted, where $k \in M(H_{k-1})$. If we choose $k \notin M(H_{k-1})$ and set $x_k = 0$, it will force a variable x_t $t \in M(H_{k-1})$ to be 0, which is set to 1 currently, and so the resulting solution set is empty. Next we set its coefficient γ_k to $d^+_{H_{k-1}}(k) - 1$ and do the aggregation with respect to the node. In this process, the newly introduced node should be a leaf in the aggregated graph H_k. Note that $\bigcup_{I \in L(H)_{k-1}} I = C \cup B_{k-1}$, where B_{k-1} is the set of variables lifted up to step $k-1$. Moreover, note that if a variable in $R_2(C)$ is not lifted so far, it cannot be in the set of leaves. Hence at any step k, if H_{k-1} has no arc, all of its nodes should be leaves, which implies that we have completed lifting on all of the variables in $R_2(C)$. Note that the procedure can be implemented in $O(|A(G)|)$ time.

Now we prove the validity of Procedure A. At any step k, general lifting requires that we should solve the following problem to determine γ_k:

$$
\begin{aligned}
\text{(LFa)}\quad z = \max \quad & \sum_{i \in C} x_i - \sum_{i \in B_{k-1}} \gamma_i x_i \\
\text{s.t} \quad & \sum_{i \in N} a_i x_i \leqslant b, && (1) \\
& x_i \geqslant x_j \qquad \text{for all } i,j \in N \text{ and } i \leqslant j \\
& x_k = 0, \\
& x_t = 1 \qquad \text{for all } t \in R_2(C)\backslash B_k, \\
& x_i \in \{0,1\} \quad \text{for all } i \in N,
\end{aligned}
$$

where $B_{k-1} = \{1, \ldots, k-1\} \subseteq R_2(C)$, that is the set of variables lifted so far. Then $\sum_{i \in C} x_i - \sum_{i \in B_k} \gamma_i x_i \leqslant |C| - 1 - \sum_{i \in B_k} \gamma_i$ is a valid inequality for $\gamma_k \leqslant |C| - 1 - \sum_{i \in B_{k-1}} \gamma_i - z$ and lifting is maximum when the equality holds. Note that in (LFa), we can set $x_i = 0$ for all $i \in N \backslash T(C)$ and $x_i = 0$ for all $i \in S_C(k)$ since $x_k = 0$. Then we can set $x_i = 0$ for all $i \in \{j \in R_1(C) | S_C(j) \subseteq S_C(k)\}$. Hence by Definition 3.2, we can discard (1). Now we can ignore the variables x_i for all $i \in R_1(C)$. After deleting the variables that are not relevant, we can transform (LFa) to the following equivalent problem:

$$
\begin{aligned}
\text{(LFb)}\quad \max \quad & \sum_{i \in C} x_i - \sum_{i \in B_{k-1}} \gamma_i x_i \\
\text{s.t} \quad & x_i \geqslant x_j \qquad \text{for all } i \in B_{k-1} \text{ and } j \in S_{C \cup B_{k-1}}(i), && (2) \\
& x_i = 0 \qquad \text{for all } i \in S_{C \cup B_{k-1}}(k), \\
& x_i \in \{0,1\} \quad \text{for all } i \in C \cup B_{k-1}.
\end{aligned}
$$

Note that the relevant set of variables $C \cup B_{k-1}$ equals $\bigcup_{I \in L(H_{k-1})} I$ and the precedence constraints in (2) occur only between the variables which belong to the same leaf in H_{k-1}. Hence, we can decompose (LFb) into the following independent subproblems with respect to each node in $L(H_{k-1})$. The following problem is the subproblem corresponding to $I \in L(H_{k-1})$.

$$\begin{aligned}
\text{(LFI)}\ \max\quad & \sum_{i \in I \cap C} x_i - \sum_{i \in I \cap B_{k-1}} \gamma_i x_i \\
\text{s.t}\quad & x_i \geqslant x_j \quad \text{for all } i \in I \cap \mathrm{B}_{k-1} \text{ and } j \in S_I(i), \\
& x_i = 0 \quad \text{for all } i \in S_I(k), \qquad (3) \\
& x_i \in \{0, 1\} \quad \text{for all } i \in I.
\end{aligned}$$

Let (LFI′) denote the problem (LFI) without (3). The following lemma characterizes the optimal solution to (LFI′).

Lemma 4.1. *The optimal objective value of* (LFI′) *is* 1 *and can be achieved only when* $x_i = 1$ *for all* $i \in I$.

Proof. The proof is given by induction on $|I \cap B_{k-1}|$. Suppose $|I \cap B_{k-1}| = 0$, the assertion clearly holds since $I = \{i\}$, for some $i \in C$. So let us assume the assertion holds for all the cases $|I \cap B_{k-1}| < l$, where $l > 0$. Let $|I \cap B_{k-1}| = l$ and $t \in I \cap B_{k-1}$ corresponds to the index of the variable lifted last among those in $I \cap B_{k-1}$. Let

$$I \backslash \{t\} = \bigcup_{j=1}^{r} I_j, \text{ where } I_j \in L(H_{t-1}) \text{ and } r = d^+_{H_{t-1}}(t).$$

Then

$$\sum_{i \in I \cap C} x_i - \sum_{i \in I \cap B_{k-1}} \gamma_i x_i = \sum_{j=1}^{r} \left(\sum_{i \in I_j \cap C} x_i - \sum_{i \in I_j \cap B_{k-1}} \gamma_i x_i \right) - (r-1) x_i,$$

since $\gamma_t = r - 1$. Since $|I_j \cap B_{k-1}| < l$ for all j, $\sum_{i \in I_j \cap C} x_i - \sum_{i \in I_j \cap B_{k-1}} \gamma_i x_i$ can have the maximum value 1 only when $x_i = 1$ for all $i \in I_j$ by the induction hypothesis. Note that if $x_t = 0$, by setting all x_i ($i \in I$) to 0, the maximum value is 0, since for each $j, j = 1, \ldots, r$, there exists at least one $i \in I_j \cap S(t)$ and so $x_i = 0$. Hence the assertion holds. □

From the above lemma, if $S_I(k) = \emptyset$, (LFI) has the maximum value 1 by setting all the variables to 1. Otherwise, since all the coefficients in the objective function are integer, the maximum value is 0 by setting all the variables to 0. Since $|\{I \in L(H_{k-1}) | S_I(k) \neq \emptyset\}| = d^+_{H_{k-1}}$, we can obtain the following proposition.

Proposition 4.2. *The optimal value of (LFb) is* $|L(H_{k-1})| - d^+_{H_{k-1}}(k)$.

Theorem 4.3. *Procedure A is valid for any IC inequality and gives a maximum lifting for an MIC inequality.*

Proof. First, assume C is an MIC. The proof is given by induction on the step number in Procedure A. At step 1, the optimal value of (LFb) is $|C| - d^+_{H_0}(1)$ since

$L(H_0) = |C|$. Hence $\gamma_1 = d^+_{H_0}(1) - 1$. Now assume the procedure is valid up to step k, where $k \geqslant 1$. At step $k+1$, the optimal value of (LFb) is $|L(H_k)| - d^+_{H_k}(k+1)$.

Note that $|L(H_k)| = |L(H_{k-1})| - d^+_{H_{k-1}}(k) + 1 = |L(H_{k-1})| - \gamma_k$. So by applying the same argument recursively, we have the relation $|L(H_k)| = |L(H_0)| - \sum_{i \in B_k} \gamma_i = |C| - \sum_{i \in B_k} \gamma_i$.

Hence, $\gamma_{k+1} = |C| - 1 - \sum_{i \in B_k} \gamma_i - |L(H_k)| + d^+_{H_k}(k+1) = d^+_{H_k}(k+1) - 1$.

If C is not an MIC, we cannot discard (1) in general. However, (LFb) gives an upper bound on (LFa). $\square$

In the proof of the above theorem, we obtain the following result.

Corollary 4.4. *At any step k of Procedure A, $\sum_{i \in B_{k-1}} \gamma_i = |C| - |L(H_{k-1})|$.*

Fig. 3 exemplifies Procedure A when applied to the MIC inequality associated to the MIC shown in figure 1, where lifting is applied in the order 3, 4, 5, 1, 2. Note that if we apply Procedure A in the order 4, 5, 3, 1, 2, we can obtain the following lifted inequality:

$$x_6 + x_7 + x_8 + x_9 - 2x_4 - x_5 \leqslant 0.$$

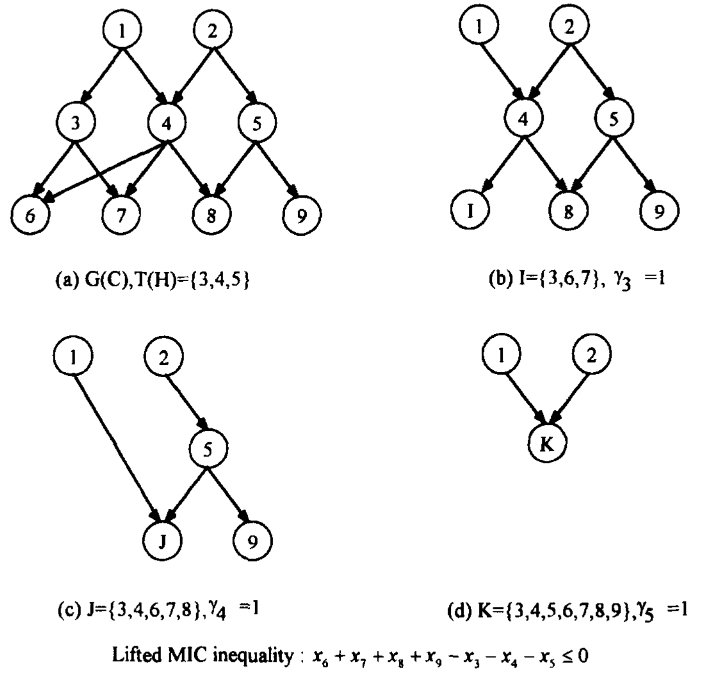


Fig. 3. Example of Procedure A.

Finally in this section, we mention that Procedure A can be applied in more general cases. For example, consider the problem of finding a maximum-weight subset in an independence system (see [10]), which contains the knapsack problem as a special case. Further assume that partial orders are given on the set of elements. Consider the *rank inequality* corresponding to a minimal dependent set (see [10]), which is a generalization of the minimal cover inequality. Similarly to Definition 3.1, we can modify it by using the fact that the precedence constraints are present. Then Procedure A can be applied to further strengthen the modified inequality. We believe that further applications of Procedure A (or some variants of it) can be found in other situations.

5. Properties of the lifted inequality

Let the connected components of $G(C)$ be $G(C_k) = (C_k \cup R_2(C_k), A(C_k))$ $k = 1, \ldots, \omega(C)$, where $\omega(C)$ is the number of connected components of $G(C)$ and $C = \bigcup_{k=1}^{\omega(C)} C_k$. Note that by Corollary 4.4, at the end of Procedure A, we have $\sum_{i \in R_2(C)} \gamma_i = |C| - \omega(C)$. Hence, at the end of Procedure A, the lifted inequality is of the form

$$\sum_{k=1}^{\omega(C)} \left(\sum_{i \in C_k} x_i - \sum_{i \in R_2(C_k)} \gamma_i x_i \right) \leqslant \omega(C) - 1.$$

Note that (LFb) can be decomposed into $\omega(C)$ independent subproblems, where each subproblem corresponds to each component of $G(C)$. So we can apply Procedure A component by component, which results in the same inequality. Hence, by Corollary 4.4, we have $\sum_{i \in R_2(C_k)} = |C_k| - 1$ for all k, $k = 1, \ldots, \omega(C)$.

We now introduce a problem which will be considered later to develop a lifting heuristic on the variables in $N \backslash T(C)$. Consider the following problem:

$$\begin{aligned} \text{(LFk)} \quad \max \quad & \sum_{i \in C_k} x_i - \sum_{i \in R_2(C_k)} \gamma_i x_i \\ \text{s.t} \quad & x_i \geqslant x_j \quad \text{if } i \leqslant j, \\ & x_i \in \{0, 1\} \quad \text{for all } i \in T(C_k) \end{aligned}$$

for each k, $k = 1, \ldots, \omega(C)$. Note that (LFk) has almost the same form as (LFI′) except for the fact that all of the variables in $R_1(C_k)$ are also included here. By similar argument used in the proof of Lemma 4.1, we can show that the optimal objective value of (LFk) is 1 and can be achieved only when all the variables have the value 1.

Proposition 5.1. *The optimal value of* (LFk) *is* 1 *and the value is achieved if and only if all of the variables are* 1.

Proof. If $x_i = 1$ for all $i \in R_1(C_k)$, (LFk) has the same form as (LFI′). Hence in this case, the result can be deduced directly from Lemma 4.1. If $x_i = 0$ for all

$i \in D \subseteq R_1(C)$. Then (LFk) has the same form as (LFI) with $S_I(k) \neq \emptyset$. Hence by similar method, we can show that the optimal value is 0 in this case. $\square$

The above proposition will be used later to develop a lifting heuristic on the variables in $N \backslash T(C)$ in Section 8.

In the following, we consider the strength of the lifting. At any step k in the lifting procedure, let x^* be the optimal solution to (LFb). From the results in the previous section,

$$x_i^* = \begin{cases} 0 & \text{if } i \in T_k, \\ 1 & \text{if } i \in (C \cup R_2(C)) \backslash T_k, \end{cases}$$

where $T_k = \{k\} \cup L_k$ and $L_k = \bigcup\{I \in L(H_k) | S_I(k) \neq \emptyset\}$. Let us extend the solution to the whole set of variables by setting $x_i^* = 0$, for all $i \in N \backslash T(C)$. We also set

$$x_i^* = \begin{cases} 0 & \text{if } S_C(i) \subseteq L_k, \\ 1 & \text{otherwise} \end{cases}$$

for $i \in R_1(C)$.

Let S_k be the set of indices of the variables which are 1 in x^*. Note that, in general, for a (not necessarily minimal) IC, C, if $a(S_k) \leqslant b$ holds at step k, the extended solution x^* is feasible to (LFa) and the lifting is maximum.

In the previous section, we show that the MIC inequality lifted by Procedure A defines a facet of $P(C)$ if the condition in Proposition 3.3 holds. However, in general, the condition is not necessary. There are some important cases where the lifted MIC inequality defines a facet of $P(C)$, though the condition does not hold. In the following, we present necessary and sufficient conditions for an MIC inequality lifted by Procedure A to define a facet of $P(C)$. Specifically, we show that for an MIC C, if $G(C)$ is connected ($\omega(C) = 1$), the lifted MIC inequality always defines a facet of $P(C)$ and if not ($\omega(C) > 1$), much weaker condition than that in Proposition 3.3 should hold.

Theorem 5.2. *For an MIC C, suppose $G(C)$ is connected, then the lifted MIC inequality*

$$\sum_{i \in C} x_i - \sum_{i \in R_2(C)} \gamma_i x_i \leqslant 0 \tag{4}$$

is facet-defining for $P(C)$.

Proof. Let $\pi x \leqslant \pi_0$ be a facet-defining inequality for $P(C)$ which contains all equality solutions to (4). We will show that the inequality can be obtained by multiplying some nonnegative constant to (4), which shows that (4) is facet-defining for $P(C)$. Since 0 is feasible and satisfies (4) at equality, $\pi_0 = 0$. For any $i \in R_1(C)$, the following solution satisfies (4) at equality:

$$x_j^i = \begin{cases} 1 & \text{if } j \in T(i), \\ 0 & \text{otherwise.} \end{cases}$$

It can be easily shown that the solutions x^i ($i \in R_1(C)$) are linearly independent. Hence, it follows $\pi_i = 0$ for all $i \in R_1(C)$. For any $i \in C$, the following solution is feasible and satisfies (4) at equality (see Definition 3.2 and Corollary 4.4):

$$x_j = \begin{cases} 1 & \text{if } j \in T(C \backslash \{i\}), \\ 0 & \text{otherwise.} \end{cases}$$

By noting that $R_2(C) \subseteq R(C\backslash\{i\})$ for all $i \in C$, and $\pi_i = 0$ for all $i \in R_1(C)$, we obtain the following result:

$$\pi_i = \pi_j \quad \text{for all } i, j \in C.$$

Hence, we can assume the equality is of the form

$$\sum_{i \in C} x_i - \sum_{i \in R_2(C)} \pi_i x_i = 0. \tag{5}$$

Moreover, by substituting the above solution into (5), we obtain

$$\sum_{i \in R_2(C)} \pi_i = |C| - 1. \tag{6}$$

In the following, we will show that $\pi_i = \gamma_i$ holds for all $i \in R_2(C)$. Let $R_2(C) = \{1, \ldots, q\}$, where $q = |R_2(C)|$ and the lifting be applied from 1 to q. The proof is given by induction on the step number in Procedure A. To show the result, we use the optimal solutions to (LFb) extended to include those variables in $R_1(C)$. The extension can be done similarly as above and the extended solution is clearly feasible to (LFa). Hence, the extended solution satisfies (4) at equality. For simplicity of presentation, we ignore the values of x_i for $i \in R_1(C)$.

At step 1 in Procedure A, the optimal solution to (LFb) is

$$x_j = \begin{cases} 0 & \text{if } j \in \{1\} \cup S_C(1), \\ 1 & \text{otherwise.} \end{cases}$$

By substituting the solution into (5), we obtain

$$\sum_{i=2}^{q} \pi_i = |C| - |S_C(1)|.$$

The above equation with (6) gives $\pi_1 = |S_C(1)| - 1 = d^+_{H_0}(1) - 1 = \gamma_1$. Now, let us assume $\pi_i = \gamma_i$ for all $i = 1, \ldots, k-1$, where $k \geqslant 2$. At step k in Procedure A, let $L(H_{k-1}) = \{I_1, \ldots, I_l\}$, where $l = |L(H_{k-1})|$. In addition, let $d^+_{H_{k-1}}(k) = r$ and $I_i \cap S(k) \neq \emptyset$ for $i = 1, \ldots, r$. Moreover, let

$$I_i \cap B_{k-1} \begin{cases} \neq \emptyset & \text{for } i = r+1, \ldots, s, \\ = \emptyset & \text{for } i = s+1, \ldots, l, \end{cases}$$

where $B_{k-1} = \{1, \ldots, k-1\}$.

Then for $i = r+1, \ldots, s$, $\sum_{j \in I_i \cap B_{k-1}} \pi_j = \sum_{i \in I_i \cap B_{k-1}} \gamma_j = |I_i \cap C| - 1$ holds (it can be easily deduced from Corollary 4.4). For $i = s+1, \ldots, l$, $|I_i \cap C| = 1$ since

$I_i = I_i \cap C = \{j\}$ for some $j \in C$. The optimal solution to (LFb) at step k is

$$x_j = \begin{cases} 0 & \text{if } j \in \{k\} \cup \bigcup_{i=1}^{r} I_i, \\ 1 & \text{otherwise.} \end{cases}$$

We prove the result in two cases. First, suppose $r = s$ holds and so $x_i = 0$ for all $i \in B_{k-1}$. By substituting the above solution into (5), we obtain $\sum_{i=k+1}^{q} \pi_i = |C| - \sum_{i=1}^{r} |I_i \cap C|$. By noting that

$$|C| - \sum_{i=1}^{r} |I_i \cap C| = \sum_{i=r+1}^{l} |I_i \cap C| = l - r,$$

we obtain $\sum_{i=k+1}^{q} \pi_i = l - r$ and by Corollary 4.4, $\sum_{i=1}^{k-1} \pi_i = \sum_{i=1}^{k-1} \gamma_i = |C| - l$. These two equations with (6) establish $\pi_k = r - 1 = d_{H_k}^{+}(k) - 1 = \gamma_k$.
Second, suppose $r < s$ holds. Then by substituting the above solution into (5), we obtain

$$\sum_{i=k+1}^{q} \pi_i = |C| - \sum_{i=1}^{r} |I_i \cap C| - \sum_{i=r+1}^{s} \sum_{j \in I_i \cap B_{k-1}} \gamma_j.$$

As noted above,

$$\sum_{i=r+1}^{s} \sum_{j \in I_i \cap B_{k-1}} \gamma_j = \sum_{i=r+1}^{s} (|I_i \cap C| - 1) = \sum_{i=r+1}^{s} |I_i \cap C| - s + r.$$

So

$$\sum_{i=k+1}^{q} \pi_i = |C| - \sum_{i=1}^{s} |I_i \cap C| + s - r = \sum_{i=s+1}^{l} |I_i \cap C| + s - r = l - r.$$

Hence as in the first case, we can show $\pi_k = \gamma_k$. $\square$

For example, consider the MIC C shown in Fig. 2, where $R_2(C) = \{2, 4\}$. Note that $G(C)$ is connected. By choosing 2 first, we obtain $x_6 + x_7 + x_8 - x_2 - x_4 \leqslant 0$ and by choosing 4 first, we obtain $x_6 + x_7 + x_8 - 2x_4 \leqslant 0$. Both inequalities define facets of $P(C)$.

Now, we consider the case $G(C)$ is not connected, that is, $\omega(C) > 1$. In this case, the lifted MIC inequality does not always define a facet of $P(C)$. To do so, the following condition should hold. For all $j \in R_1(C_k)$, where $k = 1, \ldots, \omega(C)$,

$$a(T(C \backslash C_k)) + a(T(j)) \leqslant b. \tag{7}$$

Note that the condition (7) can be checked easily. Condition (7) is similar to the condition in Proposition 3.3. Note that $a(T(C\backslash\{i\}))$ in Proposition 3.3 is replaced with $a(T(C\backslash C_k))$, where $i \in C_k$ and $j \leqslant i$. Note that for any equality solution x to the lifted MIC inequality

$$\sum_{k=1}^{\omega(C)} \left(\sum_{i \in C_k} x_i - \sum_{i \in R_2 C_k} \gamma_i x_i \right) \leqslant \omega(C) - 1, \tag{8}$$

$x_i = 1$ for all $i \in T(C_k)$ and $k \in K$ should hold, where $K \subseteq (1, \ldots, \omega(C)\}$ and $|K| = \omega(C) - 1$ (see Proposition 5.1). This fact gives a partial explanation why the condition in Proposition 3.3 can be replaced with (7). The next theorem addresses the significance of $G(C)$ being connected.

Theorem 5.3. *For an MIC C, if $G(C)$ is not connected, the lifted inequality* (8) *defines a facet of $P(C)$ if and only if the condition* (7) *holds.*

Proof. We prove only the necessity part. The sufficiency of condition (7) can be proved in a similar way as in the proof of Theorem 5.2. The result is proved by showing that if the condition (7) does not hold, the inequality (8) is a sum of some subset of precedence constraints and a valid inequality for $P(C)$.

Suppose the condition (7) does not hold for some $j \in R_1(C_k)$. Then $C' = (C \backslash C_k) \cup \{j\}$ is an IC. Note that $R_2(C') = R_2(C) \backslash R_2(C_k)$. Let us apply Procedure A to the IC inequality $x(C') \leqslant |C'| - 1$ by choosing the variables in $R_2(C')$ in the same (relative) order used when lifting the inequality $x(C) \leqslant |C| - 1$. Let the resulting inequality be

$$\sum_{i \in C'} x_i - \sum_{i \in R_2(C')} \gamma_i' x_i \leqslant \omega(C') - 1. \tag{9}$$

Note that $\omega(C') = \omega(C)$ and $\gamma_i' = \gamma_i$ for all $i \in R_2(C')$. Hence, the inequality (9) is of the same form as (8) except that the variables in $C_k \cup R_2(C_k)$ do not appear and x_j is included. Note that $\sum_{i \in R_2(C_k)} \gamma_i = |C_k| - 1$ holds and $\sum_{i \in D} \gamma_i \leqslant |S_{C_k}(D)| - 1$ for all $D \subseteq R_2(C_k)$ and $D \neq \emptyset$ (see the next section). In the following, we show that (8) can be obtained by adding some subset of precedence constraints to the inequality (9). To do so, first, let us consider a transportation problem (TP) to determine which of the precedence constraints are needed. Let $A = C_k$, the set of supplies with amount of supply $c_i = 1$ for all $i \in A$ and $B = \{j\} \cup R_2(C_k)$, the set of demands with amount of demand $d_i = 1$ for $i = j$ and $d_i = \gamma_i$ for all $i \in R_2(C_k)$. Note that $c(A) = d(B)$. An arc (a, b), where $a \in A$, $b \in B$, exists if and only if $b \leqslant a$. We can show the problem is feasible by the well-known *max-flow mincut* theorem. Note that for any subset $D \subseteq B$, if $j \in D$ and $D \cap R_2(C) \neq \emptyset$,

$$\begin{aligned} d(D) &= \gamma(D \cap R_2(C_k)) + 1 \\ &\leqslant |S_{C_k}(D)| = c(A(D)), \end{aligned}$$

where $A(D) = \{a \in A \mid (a, i) \text{ exists for some } i \in D\}$.

When $j \notin D$, similar result holds. Let us assume a feasible solution to (TP) be given. For each $i \in A$, let $(i, b(i))$ be the (unique) arc chosen in the solution. Then by adding the subset of precedence constraints $x_i - x_{b(i)} \leqslant 0$ for all i, C_k to (9), we obtain (8). $\square$

6. Choice of lifting order

In this section, we consider the problem of finding an optimal order of lifting. Formally, the problem is stated as follows.

(FL) *Given a fractional solution x^* and an IC C, either show that there is no lifted IC inequality that is violated by the given solution, or find one which is most violated by the solution.*

This problem is very important if we want to use the lifted IC inequality as a cutting plane in solving integer programming problems. In practice, one may choose the lifting order on the basis of the solution value x_i^* *for* $i \in R_2(C)$ or on the value $\gamma_i x_i^*$ for $i \in R_2(C)$. At first glance, the latter choice seems more desirable since it considers the solution value and lifted coefficient at the same time. However, in the following, we show that a simple method based on the solution value gives an optimal choice. Since the lifted inequality is of the form

$$\sum_{i \in C} x_i - \sum_{i \in R_2(C)} \gamma_i x_i \leqslant \omega(C) - 1,$$

the problem (FL) can be solved by solving the following problem:

$$\text{(FL)} \quad \min \quad \sum_{i \in R_2(C)} \gamma_i x_i^*$$

$$\gamma \in \Gamma,$$

where Γ is the set of $|R_2(C)|$-vectors composed of possible lifting coefficients obtained by applying Procedure A. The result in this section shows that the following simple procedure gives an optimal choice of lifting. First, sort the solution values. Let the result be as follows:

$$x_1^* \leqslant x_2^* \leqslant \cdots \leqslant x_q^*,$$

where $q = |R_2(C)|$. Then apply Procedure A from 1 to q. Let us call such an order of lifting *a greedy order*. We mention that the greedy order will never conflict with the requirement of Procedure A in the order in which the variables must be chosen due to the precedence constraints. More comments on the fact will be given in the end of this section.

Now, we will give the proof. First, let us assume that $G(C)$ is bipartite, which itself appears in many important applications [5, 7, 9]. The case where $G(C)$ is not bipartite will be considered later. For notational convenience, let D denote the set $R_2(C)$. For a given $A \subseteq D$, the aggregated graph with respect to A denotes the graph obtained by applying sequentially the aggregation to $G(C)$ with respect to $i \in A$. Note that the resulting graph is unique up to isomorphism independent of the order of aggregation. Let $l(A)$ denote the number of leaves in the graph obtained. Let f be a set-function defined on all subsets of D as follows.

$$f(A) = |C| - l(A) \quad \text{for all } A \subseteq D. \tag{10}$$

Then $f(A)$ is the sum of lifted coefficients of x_i, $i \in A$ which are obtained if we apply Procedure A first to the set A (see Corollary 4.4). Note that l is a monotone decreasing set-function, i.e.,

$$l(A) \geqslant l(B) \quad \text{if } A \subseteq B \subseteq D.$$

Hence, $f(A)$ is monotone increasing. A set-function g defined on all subsets of D is called *submodular* if

$$g(A) + g(B) \geqslant g(A \cup B) + g(A \cap B) \quad \text{for all } A, B \subseteq D.$$

An alternative characterization of submodularity is the following.

Proposition 6.1 (Lova'sz [8]). *Let g be a function defined on all subsets of D. Then g is submodular if and only if the derived set-functions*

$$g_a(X) = g(X \cup \{a\}) - g(X), \quad X \subseteq D\backslash\{a\}$$

are monotone decreasing for all $a \in D$.

By using the above proposition, we can show that the function f defined by (10) is submodular.

Proposition 6.2. *The function f defined by* (10) *is submodular and nondecreasing.*

Proof. Let $A \subseteq B \subseteq D\backslash\{a\}$, where $a \in D$. Then

$$f_a(A) = l(A) - l(A \cup \{\mathrm{a}\}) \equiv \gamma_a^1,$$

$$f_a(B) = l(B) - l(B \cup \{\mathrm{a}\}) \equiv \gamma_a^2.$$

Note that $\gamma_a^1(\gamma_a^2)$ is the coefficient of x_a obtained when we apply Procedure A first to the set $A(B)$ and then to a. It can be easily shown that $\gamma_a^1 \geqslant \gamma_a^2$ (or it can be deduced from the properties of sequential lifting, see [10]). $\square$

Next, consider the following polytope:

$$P_\gamma(C) = \{\gamma \in R_+^q \,|\, \gamma(A) \leqslant f(A) \text{ for all } A \subset D,\ \gamma(D) = f(D)\},$$

where $\gamma(A) = \sum_{i \in A} \gamma_i$ for all $A \subseteq D$. Note that $P_\gamma(C)$ is a face of the *polymatroid* [10] defined by f. It can be easily shown that $\Gamma \subseteq P_\gamma(C)$. In the following, we show that (FL) can be solved by solving the following linear program:

$$\text{(FLP)}\ \min \sum_{i \in D} c_i \gamma_i$$

$$\gamma \in P_\gamma(C),$$

by setting $c_j = x_j^*$.
Suppose $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_q$ and $A^i = \{1, \ldots, i\}$ for $i \in D$ with $A^0 = \emptyset$.

Proposition 6.3. *An optimal solution to* (FLP) *is*

$$\gamma_i = f(A^i) - f(A^{i-1}) \quad \textit{for } i \in D.$$

Proof. The proposed solution is primal feasible, because f nondecreasing implies $\gamma_i \geqslant 0$, and because we have for all $A \subseteq D$,

$$\begin{aligned}
\gamma(A) &= \sum_{i \in A} \gamma_i \\
&= \sum_{i \in A} [f(A^i) - f(A^{i-1})] \\
&\leqslant \sum_{i \in A} [f(A^i \cap A) - f(A^{i-1} \cap A)] \quad \text{(by the submodularity of } f) \\
&\leqslant f(A^q \cap A) - f(\emptyset) \\
&= f(A) - f(\emptyset) = f(A).
\end{aligned}$$

Let us consider the dual of (FLP):

$$\begin{aligned}
\min \quad & \sum_{A \subset D} f(A) y_A + f(D) y_D \\
\text{s.t.} \quad & \sum_{A: i \in A} y_A + y_D \leqslant c_i \quad \text{for all } i \in D, \\
& y_A \leqslant 0 \quad \text{for all } A \subset D, \\
& y_D \text{ unrestricted in sign.}
\end{aligned}$$

We show that the following solution is feasible to the above LP:

$$y_{A^i} = c_i - c_{i+1} \quad \text{for } i = 1, \ldots, q-1,$$

$$y_{A^q} = c_q,$$

$$y_A = 0, \text{ otherwise.}$$

Note that $y_A \leqslant 0$, for all $A \neq A_q$ and $\sum_{A: i \in A} y_A + y_D = y_{A^i} + \cdots + y_{A^q} = c_i$. Hence, the solution is dual feasible. The primal objective value is $\sum_{i \in D} c_i [f(A^i) - f(A^{i-1})]$,

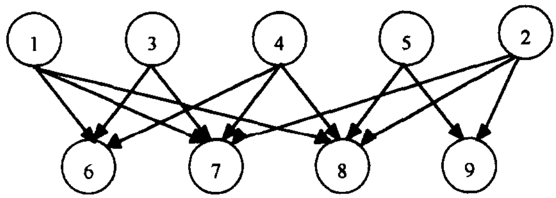


Fig. 4.

and the dual objective value is

$$\sum_{i=1}^{q-1} f(A^i)(c_i - c_{i+1}) + f(A^q)c_q = \sum_{i \in D} c_i[f(A^i) - f(A^{i-1})].$$

Hence the proposed solution is optimal. $\square$

Note that the solution in Proposition 6.3 can be obtained by applying Procedure A from 1 to q. Hence, for any instance of (FLP), on optimal solution can be found in Γ, which shows the following corollary.

Corollary 6.4. *If $G(C)$ is bipartite,* $\text{conv}(\Gamma) = P_\gamma(C)$.

Now we consider the case $G(C)$ is not bipartite. By removing all the arcs (i,j) such that $i,j \in D$, we can obtain a bipartite graph (for example, see Fig. 4, for the case of MIC N shown in Fig. 1). Then we can apply the above result to the resulting bipartite graph. In this case, note that the lifting order chosen may not be a feasible sequence in applying Procedure A. However, this problem can be overcome easily by noticing that if there is an arc (i,j) for $i,j \in D$, then $x_i^* \geqslant x_j^*$. Hence, when applying the result of Proposition 6.3, we can choose a feasible lifting order which also satisfies the conditions of the proposition. So in any case, the greedy lifting order is optimal.

Theorem 6.5. *For a given induced cover C and a fractional solution x^*, the greedy order of lifting gives a lifted cover inequality corresponding to an optimal solution to (FL).*

7. The vertices of P_{LP}

In this section, we consider the LP relaxation of (PK). Let us denote the polytope defined by this LP relaxation as P_{LP}, where

$$P_{\mathrm{LP}} = \left\{ x \in R^n \,\middle|\, \sum_{i \in N} a_i x_i \leqslant b,\ x_i \leqslant x_j,\ \text{if } i \leqslant j,\ 0 \leqslant x_i \leqslant 1, \text{ for all } i \in N \right\}.$$

Let us call a vertex x^* of P_{LP} *fractional* if it has a coordinate of fractional value, otherwise, *integral*. In the following, we show that each fractional vertex of P_{LP} has the fractional coordinates of the same value. Moreover, it is shown that the subgraph of G induced by the nodes corresponding to the fractional coordinates is connected. Using these results, we show that all fractional vertices can be cut off by IC inequalities lifted by applying Procedure A.

Theorem 7.1. *Every vertex x^* of P_{LP} has the following form:*

$$x_i^* = \eta \quad \textit{for all } i \in F,$$

where F is the set of coordinates which have fractional values. Moreover, the subgraph of G induced by F is connected.

Proof. Let x^* be a fractional vertex of P_{LP}. Then there exists a system of linear equations which has x^* as a unique solution. Note that $\sum_{i \in N} a_i x_i^* = b$ holds, since otherwise, the remaining constraints constitute an integral polyhedron [1]. After substituting the variables which have integral values, we can obtain a system of linear equations of the following form.

$$\text{(SLE)} \quad \sum_{i \in F} a_i x_i = b' \tag{11}$$

$$x_i = x_j \quad \text{for all } (i, j) \in A(F), \tag{12}$$

where $b' = b - \sum_{i \in N \setminus F} a_i x_i^*$ and $A(F) \subseteq A \cap (F \times F)$. We can assume that the unique solution of (SLE) is x_j^*, $j \in F$. Suppose $A(F) = \emptyset$, then (SLE) consists of only (11). In this case, the assertion trivially holds since $|F| = 1$. So in the following, we assume $A(F)$ is not empty. Suppose there exists a variable which only appears in (11), we can discard the equation since the value of the variable can be determined uniquely by the other variables. In this case, the resulting system consists of only (12) and so the unique solution is 0, which leads to contradiction. Hence, all variables in F should appear at least in one equation in (12). By using the equations in (12), we can partition F into the subsets F_t, where $t = 1, \ldots, r$ such that the variables in the same subset should have the same value. Moreover, choose the partition such that r is as small as possible. Next substitute the set of equations in (12) by the following:

$$x_i = x_t \quad \text{for all } i \in F_t,\ t = 1, \ldots, r.$$

Then the system can be reduced to the following single equation:

$$\sum_{t=1}^{r} \left(\sum_{j \in F_t} a_j \right) x_i = b',$$

which should have unique solution. Hence $r = 1$ and $\eta = (b - \sum_{i \in N \setminus F} a_i) / \sum_{i \in F} a_i$. □

Corollary 7.2. *For each fractional vertex x^* of P_{LP}, there exists a lifted IC inequality which is violated by x^*.*

Proof. For a given fractional vertex x^*, let $T(C) = F \cup N_1$, where N_1 is the set of variables of value 1 and C is the set of leaves in the subgraph of G induced by $F \cup N_1$. Note that $a(T(C)) > b$, and hence C is an IC. Let us apply Procedure A first on the set $F \cap R_2(C)$. Then $\sum_{i \in F \cap R_2(C)} \gamma_i = |F \cap C| - 1$ since the subgraph of $G(C)$ induced by F is connected. Let the resulting IC inequality be as follows:

$$\left(\sum_{i \in C \cap F} x_i - \sum_{i \in R_2(C) \cap F} \gamma_i x_i \right) + \left(\sum_{i \in C \cap N_1} x_i - \sum_{i \in R_2(C) \cap N_1} \gamma_i x_i \right) \leqslant \omega(C) - 1.$$

Since $|F \cap C| \neq \emptyset$, the first term in the above inequality when substituted by x^*, is η and the second term is less than or equal to $\omega(C) - 1$ (see Proposition 5.1). Hence, the inequality is violated by the given fractional vertex at least by η. $\square$

8. Lifting heuristics on the other variables

In this section, we consider lifting of the inequality obtained by applying Procedure A on the variables x_i $(i \in N \backslash T(C))$. When $\omega(C) = 1$, if $R(j) \cap (B \cup C) = \emptyset$, where $j \in N \backslash T(C)$ and $B = \{i \in R_2(C) \mid \gamma_i > 0\}$, then it can be easily shown that lifting on x_j results in 0 coefficient. When $\omega(C) > 1$, if there exists $k \in \{1, \ldots, \omega(C)\}$ such that $R(j) \cap T(C_k) = \emptyset$ and $a(T(C \backslash C_k) \cup R(j)) \leqslant b$, then lifting on x_j also results in 0 coefficient. In general, lifting problem on x_i $(i \in N \backslash T(C))$ is a PK.

In the following, we give a general lifting heuristic on the variables in $N \backslash T(C)$. The proposed heuristic procedure is an extension of the sequential lifting procedure for the cover inequality of the knapsack polytope [10]. Let the currently lifted variables be x_i $i \in A_k$, where $A_k = \{1, \ldots, k\} \subseteq N \backslash T(C)$ and the lifted inequality is given as follows:

$$\sum_{j=1}^{\omega(C)} \left(\sum_{i \in C_j} x_i - \sum_{i \in R_2(C_j)} \gamma_i x_i \right) + \sum_{i \in A_k} \alpha_i x_i \leqslant \omega(C) - 1.$$

Consider the following knapsack problem.

$$\begin{aligned} \text{(LFc)} \quad \eta_{k+1} = \max \quad & \sum_{i=1}^{\omega(C)} z_i + \sum_{i \in A_k} \alpha_i x_i \\ & \sum_{i=1}^{\omega(C)} w_i z_i + \sum_{i \in A_k} a_i x_i \leqslant b - a_{k+1}, \\ & z_i \in \{0, 1\} \quad \text{for all } i = , \ldots, \omega(C), \\ & x_i \in \{0, 1\} \quad \text{for all } i \in A_k, \end{aligned}$$

where $w_i = a(T(C_i))$.

In (LFc), z_i corresponds to the aggregation of all of the variables corresponding to $T(C_i)$. From the results of Proposition 5.1, (LFc) can be viewed as a relaxation of the problem which should be solved to obtain the lifted coefficient of x_{k+1}. Only the inequalities $x_i \geqslant x_j$, where i, j are contained in the same component of $G(C)$, are implicitly considered here (in the form of the aggregated variables corresponding to the components). All the other inequalities are ignored. Hence, η_{k+1} gives an upper bound on the true optimal value. So, if we set the coefficient of x_{k+1} in the lifted inequality (α_{k+1}) to $\omega(C) - 1 - \eta_{k+1}$, the resulting inequality is valid. Hence, by solving the corresponding (LFc), we can lift (approximately) on the variables x_i for $i \in N \backslash T(C)$. Note that this procedure coincides with the usual lifting procedure for the cover inequality on the variables x_i $i \in N \backslash C$ when there are no precedence constraints; see [10].

9. Concluding remarks

This paper considers the polyhedral structure of the precedence-constrained knapsack problem. We propose some modifications on the notion of the cover inequality. A combinatorial lifting procedure which runs in $O(|A(G)|)$ time is presented. Within our knowledge, few lifting procedures as easy as this have appeared in the literature. We also propose another lifting procedure to further strengthen the inequality. Properties of the lifted cover inequalities are considered and the problem of finding an optimal order of lifting is solved.

The results in this paper can be used in devising a cutting-plane algorithm for a problem which contains (PK) as a substructure. Computational study [11] applied to the problem in [9] indicates the usefulness of the lifted IC inequality. In [11], only partial implementation of the results given in this paper provided significant improvement in the performance of the algorithm. Since the precedence-constrained knapsack structures appear in many important integer programming applications, we believe that the results presented in this paper can be helpful in solving such problems using strong cutting plane approach.

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