ELSEVIER

European Journal of Operational Research 91 (1996) 528–542

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Theory and Methodology

Joint price and lot size determination under conditions of permissible delay in payments and quantity discounts for freight cost

Seong Whan Shinn [a], Hark Hwang [b,*], Sung Soo Park [b]

[a] Department of Business Administration, Semyung University, Sinwol-dong, Jecheon 390-230, South Korea
[b] Department of Industrial Engineering, Korea Advanced Institute of Science and Technology, 373-1 Gusung-dong, Yusung-gu, Taejon 305-701, South Korea

Received September 1993; revised December 1994

Abstract

This paper deals with the problem of determining the retailer's optimal price and lot size simultaneously under conditions of permissible delay in payments. It is assumed that the ordering cost consists of a fixed set-up cost and a freight cost, where the freight cost has a quantity discount offered due to the economies of scale. The constant price elasticity demand function is adopted, which is a decreasing function of retail price. Investigation of the properties of an optimal solution allows us to develop an algorithm whose validity is illustrated through an example problem.

Keywords: Inventory; Credit period; Discounted freight cost; Pricing; Lot size

1. Introduction

In deriving the economic order quantity (EOQ) formula, it is tacitly assumed that the retailer must pay for the items as soon as he receives them from a supplier. However, in practice, a supplier will allow a certain fixed period (credit period) for settling the amount the retailer owes to him for the items supplied.

Recently, a number of research articles appeared which deal with the EOQ problem under a fixed credit period (Goyal [4], Haley and Higgins [6]). With the average cost approach, they reported that the EOQ is invariant to the length of credit period. This is not consistent with our expectation. This inconsistency is resulted by the assumption commonly held by the previous research works in which the demand for the product is treated as a given constant. Consequently, they disregarded the effects of credit period on the quantity demanded. As implicitly stated by Mehta [10], a major reason for the

* Corresponding author.

0377-2217/96/$15.00
SSDI 0377-2217(94)00357-2

supplier to offer a credit period to the retailers is to stimulate the demand for the product he produces. The supplier usually expects that the profit increases due to rising sales volume can compensate the capital losses incurred during the credit period. The positive effects of credit period on the product demand can be integrated into the EOQ model through the consideration of retailing situations where the demand rate is a function of the selling price the retailer sets for the product. The availability of the credit period from the supplier enables the retailer to choose the selling price from a wider range of option. Since the retailer's lot size is affected by the demand rate of the product, the problems of determining the retail price and the lot-size are interdependent and must be solved simultaneously (we will call this the RPLS problem). Kunreuther and Richard [8] dealt with the RPLS problem when demand is a linear function of price and when the supplier offers no quantity discounts. Abad [1] dealt with the same problem assuming that the supplier offers all-unit quantity discounts and demand for the product is a decreasing function of price. Abad [2] also extended his model to the case of incremental quantity discounts.

This paper deals with the RPLS problem when the supplier offers a certain credit period and the demand of the product is represented by a constant price elasticity function. It is also assumed that the ordering cost of the retailer contains not only a fixed cost but also a freight cost which is a function of the lot-size. In many practical situations, the order may be delivered in unit loads, i.e., trucks, containers, pallets, boxes, etc. and a quantity discount may occur in terms of the number of unit loads due to the economy of scale. The classical quantity discount EOQ model has been extensively studied in the literature (Das [3], Tersine et al. [11], Hadley and Whitin [5], and Johnson and Montgomery [7]). Noting that all these quantity discount models analyze solely the unit purchase price discount, Lee [9] studied the EOQ model with set-up cost including a fixed cost and freight cost where the freight cost has a quantity discount.

In the next section, we formulate two kinds of mathematical models: 1) optimal lot sizing policy model with price predetermined and 2) optimal pricing and lot sizing policy model. For each model, the properties of an optimal solution are discussed and solution algorithm is given in Section 3. Numerical examples are provided in Section 4, which is followed by concluding remarks.

2. Development of the model

The assumptions of this study are essentially the same as the EOQ model except for the conditions of permissible delay and the constitution of ordering cost.

The following assumptions and notations are used:

1) The demand rate is represented by a constant price elasticity function of retail price.
2) No shortages are allowed.
3) The supplier proposes a certain credit period and sales revenue generated during the credit period is deposited in an interest bearing account with rate I. At the end of the period, the credit is settled and the retailer starts paying the capital opportunity cost for the items in stock with rate R ($R \geqslant I$).
4) The retailer pays the freight cost for the transportation of the quantity purchased where the freight cost has a quantity discount.

D: Annual demand rate, as a function of retail price (P); $D = KP^{-e}$.
K: Scaling factor (> 0).
e: Index of price elasticity (> 0).
P: Unit retail price.
C: Unit purchase cost.
Q: Order size.

N_j: j-th freight cost break quantity, $j = 1, 2, \ldots, n$, where $N_0 < N_1 < \cdots < N_n < N_{n+1}$, with $N_0 = 0$ and $N_{n+1} = \infty$.
S: Fixed ordering cost.
F_j: Freight cost for Q, $N_{j-1} < Q \leqslant N_j$, where $F_{j-1} < F_j$ and $F_{j-1}/N_{j-1} > F_j/N_j$, $j = 1, 2, \ldots, n$.
t: Credit period set by the supplier.
H: Inventory carrying cost, excluding the capital opportunity cost.
R: Capital opportunity cost (as a percentage).
I: Earned interest rate (as a percentage).

Note that the inequalities $F_{j-1} < F_j$ and $F_{j-1}/N_{j-1} > F_j/N_j$ are necessary to have some quantity discount in the freight cost for changing the order size from N_{j-1} to N_j. Thus the cost for setting an order becomes $S + F_j$ for $N_{j-1} < Q \leqslant N_j$.

The retailer's objective is to maximize the annual net profit $\Pi(P, Q)$ from the sales of the products. The annual net profit consists of the following five elements:

1) annual sales revenue $= DP$;
2) annual purchasing cost $= DC$;
3) annual inventory carrying cost $= \frac{1}{2}QH$;
4) annual ordering cost $= D(S + F_j)/Q$, for $N_{j-1} < Q \leqslant N_j$;
5) annual capital opportunity cost (refer to Goyal [4]): (i) *Case 1: $Dt \leqslant Q$* (see Fig. 1a). As products are sold, the sales revenue is used to earn interest with annual rate I during the credit period t. And the average number of products in stock earning interest during time $(0, t)$ is $\frac{1}{2}Dt$ and the interest earned per order becomes $\frac{1}{2}Dt \cdot tCI$. When the credit is settled, the products still in stock have to be financed with annual rate R. Since the average number of products during time $(t, Q/D)$ becomes $\frac{1}{2}D(Q/D - t)$, the interest payable per order can be expressed as $\frac{1}{2}D(Q/D - t)(Q/D - t)CR$.

Therefore,

annual capital opportunity cost

$$= \frac{\frac{1}{2}D(Q/D - t)(Q/D - t)CR - \left(\frac{1}{2}Dt\right) \cdot tCI}{Q/D} = \frac{D^2C(R - I)t^2}{2Q} + \frac{QRC}{2} - DCRt.$$

(ii) *Case 2: $Dt > Q$* (see Fig. 1b). For the case of $Dt > Q$, all the sales revenue is used to earn interest with annual rate I during the credit period t. The average number of products in stock earning interest during time $(0, Q/D)$ and $(Q/D, t)$ become $\frac{1}{2}Q$ and Q, respectively.

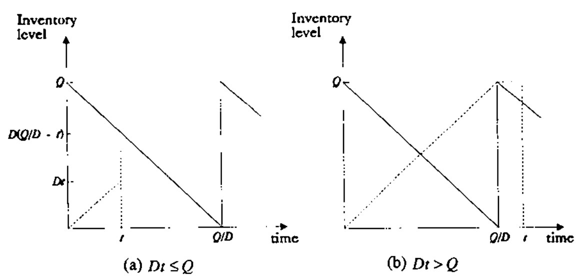

Fig. 1. Credit period vs. Q/D.

Therefore,

$$\text{annual capital opportunity cost} = -\frac{\frac{1}{2}Q(Q/D)CI + Q(t - Q/D)CI}{Q/D} = \frac{QIC}{2} - DCIt.$$

The annual net profit $\Pi(P, Q)$ can be expressed as

$$\Pi(P, Q) = \text{Sales revenue} - \text{Purchasing cost} - \text{Inventory carrying cost} - \text{Ordering cost} - \text{Capital opportunity cost}.$$

Depending on the relative size of Dt to Q, $\Pi(P, Q)$ has two different expressions, as follows:

Case 1: $Dt \leqslant Q$.

$$\Pi_{1,j}(P, Q) = DP - DC - \frac{QH}{2} - \frac{D(S + F_j)}{Q} - \left(\frac{D^2C(R - I)t^2}{2Q} + \frac{QRC}{2} - DCRt\right),$$
$$Q \in (N_{j-1}, N_j], \quad j = 1, 2, \ldots, n \tag{1}$$

Case 2: $Dt > Q$.

$$\Pi_{2,j}(P, Q) = DP - DC - \frac{QH}{2} - \frac{D(S + F_j)}{Q} - \left(\frac{QIC}{2} - DCIt\right), \quad Q \in (N_{j-1}, N_j], \quad j = 1, 2, \ldots, n. \tag{2}$$

The D^2 term in the above equations appears to be the complicating term which makes it very difficult to find an optimal solution. For the analysis, we deal with the following two models. The solution of Model 1 is to be utilized in deriving an algorithm for Model 2.

Model 1: optimal lot sizing policy model with price predetermined. Retail price is assumed to be already set by the retailer and represented by P^0. Under this model, the demand becomes constant and so the annual net profit function becomes a single variable problem of maximizing $\Pi(P^0, Q)$.

Model 2: optimal pricing and lot sizing policy model. Some retailers make pricing decision on the basis of the projected demand for their product. In this sense, the decisions with respect to price and order size represent plans rather than irrevocable commitments. Also, the availability of the credit period from the supplier tends to widen the feasible price range from which the retailer can choose an optimum retail price. Hence, in this model the pricing and lot sizing problems are considered simultaneously.

3. Determination of optimal policy

3.1. Model 1

The problem is to find an optimal lot size Q^* which maximizes $\Pi(P^0, Q)$. For the normal condition ($R \geqslant I$) as stated by Goyal [4], $\Pi_{i,j}(P^0, Q)$ is a concave function for every i and j. And so, there exists a unique value $Q_{i,j}$ which maximizes $\Pi_{i,j}(P^0, Q)$ which is given by

$$Q_{1,j} = \sqrt{2D(S_1 + F_j)/H_1}, \tag{3}$$

where $S_1 = S + \frac{1}{2}DC(R - I)t^2$ and $H_1 = H + CR$, and

$$Q_{2,j} = \sqrt{2D(S + F_j)/H_2}, \tag{4}$$

where $H_2 = H + CI$.

$Q_{i,j}$ and $\Pi_{i,j}(P^0, Q)$ can be shown to have the following properties.

Property 1. *For i given,* $Q_{i,j} < Q_{i,j+1}$, $j = 1, 2, \ldots, n-1$.

Property 2. *For any* Q, $\Pi_{i,j}(P^0, Q) > \Pi_{i,j+1}(P^0, Q)$, $i = 1, 2$; $j = 1, 2, \ldots, n-1$.

Property 1 indicates that the value of both $Q_{1,j}$ and $Q_{2,j}$ is strictly increasing as j increases. Property 2 implies that both $\Pi_{1,j}(P^0, Q)$ and $\Pi_{2,j}(P^0, Q)$ are strictly decreasing for any fixed value of Q as j increases. Since our problem structure satisfies these two properties, we are able to adopt the results of Lee [9] in developing the solution algorithm of Model 1. Now, we present two theorems, one for Case 1 and the other for Case 2. Based on these theorems, we only need to consider a finite number of candidate values of Q in finding an optimal value Q^*.

Theorem 1 (for Case 1). *Suppose Dt belongs to* $(N_{a-1}, N_a]$ *for some a. Let b be the largest index such that* $Q_{1,0} > N_b$, *where* $Q_{1,0} = \sqrt{2DS_1/H_1}$. *Also, let c be the larger value of a and b, and k* $(k \geqslant c)$ *be the first index such that* $Q_{1,k} \leqslant N_k$, *respectively.*

(i) *If the index k* $(\leqslant n)$ *exists and* $Q_{1,k} < Dt$, *then* Q^* *must be less than Dt.*

(ii) *If the index k* $(\leqslant n)$ *exists and* $Q_{1,k} \geqslant Dt$, *then we have to consider* $Q = N_c, N_{c+1}, \ldots, N_{k-1}, Q_{1,k}$ *only as candidates for* Q^*.

(iii) *If* $Q_{1,j} > N_j$ *for all* $c \leqslant j \leqslant n$, *then we have to consider* $Q = N_c, N_{c+1}, \ldots, N_n$ *as candidates for* Q^*.

Theorem 2 (for Case 2). *Suppose Dt belongs to* $(N_{a-1}, N_a]$ *for some a. Let b be the largest index such that* $Q_{2,0} > N_b$, *where* $Q_{2,0} = \sqrt{2DS/H_2}$, *and k* $(k > b)$ *be the first index such that* $Q_{2,k} \leqslant N_k$, *respectively.*

(i) *If* $N_b \geqslant Dt$, N_{a-1} *becomes the only candidate for* Q^*.

(ii) *If the index k* $(\leqslant a)$ *exists and* $Q_{2,k} < Dt$, *then we have to consider* $Q = N_b, N_{b+1}, \ldots, N_{k-1}, Q_{2,k}$ *as candidates for* Q^*.

(iii) *If* $Q_{2,j} > N_j$ *for all* $b \leqslant j < a$ *and* $Q_{2,a} \geqslant Dt$, *then we have to consider* $Q = N_b, N_{b+1}, \ldots, N_{a-1}$ *as candidates for* Q^*.

To facilitate the explanation of Theorems 1 and 2, we present Fig. 2, which shows the shape of $\Pi_{i,j}(P^0, Q)$ of the example problem introduced in Section 4.1. Note that $\Pi_{i,j}(P^0, Q)$ in solid lines satisfies both Properties 1 and 2. Applying Theorems 1 and 2 to the problem, it is found that $a = 2$, $b = 1$, $c = 2$, $k = 3$ for Case 1 $(Q \geqslant Dt)$ and $a = 2$, $b = 1$ for Case 2 $(Q < Dt)$. Hence, the candidates for an optimal Q are N_2 and $Q_{1,3}$ for Case 1 and N_1 for Case 2, and the optimal lot size becomes N_2 with its maximum annual net profit of \$8568. The contents of (ii) and (iii) in each theorem are essentially repetitions of Lee's finding [9] and so we omit the proofs. The proofs for (i) are given in the Appendix.

Based on the above theorems, we develop the following solution procedure for determining an optimal lot size Q^*.

Solution algorithm for Model 1

Step 1. This step identifies all the candidate values Q_0 of Q satisfying $Q_0 \geqslant Dt$ (for Case 1) and the corresponding annual net profit $\Pi(P^0, Q_0)$.

Step 1.1. Compute $Q_{1,0} = \sqrt{2DS_1/H_1}$ and find index b such that $Q_{1,0} \in (N_b, N_{b+1}]$.

Step 1.2. Find index a such that $Dt \in (N_{a-1}, N_a]$ and let $c = \max[a, b]$.

Step 1.3. Compute $Q_{1,j}$ by Eq. (3) and find the first index k $(k \geqslant c)$} such that $Q_{1,k} \leqslant N_k$.

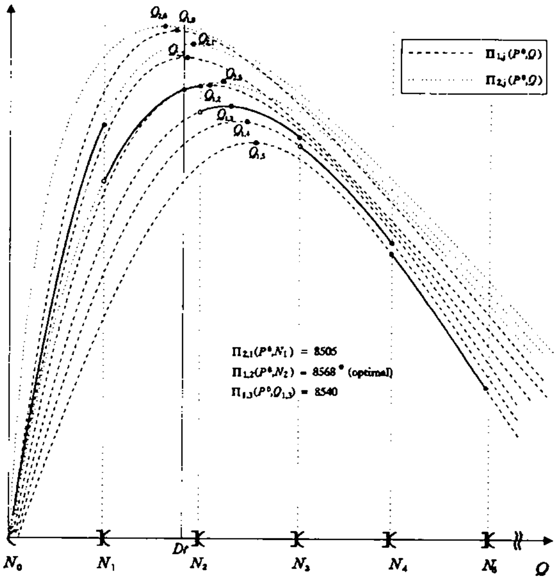

Fig. 2. $\Pi_{i,j}(P^0, Q)$ of an example problem in Section 4.1.

Step 1.4. If the index k ($\leqslant$ n) exists, then go to Step 1.5.
Otherwise, compute the annual net profit with Eq. (1) for $Q_0 = N_c, N_{c+1}, \dots, N_n$ and go to Step 2.

Step 1.5. If $Q_{1,k} < Dt$, then go to Step 2.
Otherwise, compute the annual net profit with Eq. (1) for $Q_0 = N_c, N_{c+1}, \dots, N_{k-1}, Q_{1,k}$ and go to Step 2.

Step 2. This step identifies all the candidate values Q_0 of Q satisfying $Q_0 < Dt$ (for Case 2) and the corresponding annual net profit $\Pi(P^0, Q_0)$.

Step 2.1. Compute $Q_{2,0} = \sqrt{2DS/H_2}$ and find index b such that $Q_{2,0} \in (N_b, N_{b+1}]$.

Step 2.2. If $N_b \geqslant Dt$, then compute the annual net profit with Eq. (2) for $Q_0 = N_{a-1}$ and go to Step 3.
Otherwise, compute $Q_{2,j}$ by Eq. (4) and find the first index k ($k > b$) such that $Q_{2,k} \leqslant N_k$ and go to Step 2.3.

Step 2.3. If the index k ($\leqslant a$) exists and $Q_{2,k} < Dt$, then compute the annual net profit with Eq. (2) for $Q_0 = N_b, N_{b+1}, \dots, N_{k-1}, Q_{2,k}$ and go to Step 3.
Otherwise, compute the annual net profit for $Q_0 = N_b, N_{b+1}, \dots, N_{a-1}$ and go to Step 3.

Step 3. Select the optimal lot size (Q^*) among Q_0 found in Steps 1 and 2 which gives the maximum annual net profit.

3.2. Model 2

In this model, we want to find (P^*, Q^*) which maximizes $\Pi(P, Q)$. Theorems 1 and 2 state that for $P = P^0$ fixed, only the elements in set $B = \{N_j, Q_{i,j}(P^0)$ for $i = 1, 2$ and $j = 1, 2, \dots, n\}$ become candidates for an optimal lot size $Q^*(P^0)$ where $Q_{i,j}(P^0)$ is obtained by substituting P with P^0 in Eqs. (3) and (4). Noting that some elements of B can be dropped from consideration in search of $Q^*(P)$, we formulate the following conditions $Q_{i,j}(P)$ and N_j must satisfy to become a candidate of $Q^*(P)$.

(C-1). The conditions for $Q_{i,j}(P)$ to be a candidate of $Q^*(P)$ are:

$$Q_{1,j}(P) \geqslant Dt \quad \text{and} \quad N_{j-1} < Q_{1,j}(P) \leqslant N_j \quad \text{for Case 1,} \tag{5}$$

$$Q_{2,j}(P) < Dt \quad \text{and} \quad N_{j-1} < Q_{2,j}(P) \leqslant N_j \quad \text{for Case 2.} \tag{6}$$

(C-2). The conditions for N_j to be a candidate of $Q^*(P)$ are:

$$N_j \geqslant Dt \quad \text{and} \quad N_j < Q_{1,j}(P) \quad \text{for Case 1,} \tag{7}$$

$$N_j < Dt \quad \text{and} \quad N_j < Q_{2,j}(P) \quad \text{for Case 2.} \tag{8}$$

For $Q_{i,j}(P)$ to be a candidate of $Q^*(P)$ in Case 1, $Q_{1,j}(P)$ must lie on $(N_{j-1}, N_j]$ and also $Q_{1,j}(P) \geqslant Dt$ must hold. For N_j to be a candidate of $Q^*(P)$ in Case 1, $\Pi_{1,j}(P, Q)$ must be increasing at N_j. In other words, the conditions $N_j < Q_{1,j}(P)$ and $N_j \geqslant Dt$ must be satisfied. The conditions for Case 2, Eqs. (6) and (8), are justified in a similar way.

Now, let us consider $Q_{1,j}(P) \geqslant Dt$ in Eq. (5). Since the demand rate D is also a function of P, the inequality can be written as

$$Q_{1,j}(P) = \sqrt{\frac{2D(S+F_j) + D^2C(R-I)t^2}{H+CR}} \geqslant Dt = KP^{-e}t. \tag{9}$$

Rearranging Eq. (9),

$$P \geqslant \left(\tfrac{1}{2}Kt^2(H+CI)/(S+F_j)\right)^{1/e}. \tag{10}$$

Let

$$P1_j = \left(\tfrac{1}{2}KH_2t^2/(S+F_j)\right)^{1/e}. \tag{11}$$

It is self-evident that for any $P \geqslant P1_j$, the inequality $Q_{1,j}(P) \geqslant Dt$ holds. Similarly, $N_{j-1} < Q_{1,j}(P)$ in Eq. (5) can be rewritten as

$$P < P2_j, \tag{12}$$

where

$$P2_j = \begin{cases} \left(\dfrac{KC(R-I)t^2}{\sqrt{(S+F_j)^2 + C(R-I)H_1t^2N_{j-1}^2} - (S+F_j)} \right)^{1/e} & \text{if } R > I, \\ \left(2K(S+F_j)/\left(H_1N_{j-1}^2\right)\right)^{1/e} & \text{if } R = I. \end{cases}$$

Also, from $Q_{1,j}(P) \leqslant N_j$ in Eq. (5), we have

$$P \geqslant P3_j, \tag{13}$$

where

$$P3_j = \begin{cases} \left(\dfrac{KC(R-I)t^2}{\sqrt{(S+F_j)^2 + C(R-I)H_1t^2N_j^2} - (S+F_j)} \right)^{1/e} & \text{if } R > I, \\ \left(2K(S+F_j)/\left(H_1N_j^2\right)\right)^{1/e} & \text{if } R = I. \end{cases}$$

With a similar procedure, other price ranges are obtained from inequalities in Eqs. (6), (7) and (8). They are:

$$P < P1_j, \quad \text{where } P1_j = \left(\tfrac{1}{2}KH_2t^2/(S+F_j)\right)^{1/e} \quad \text{from } Q_{2,j}(P) < Dt, \tag{14}$$

$$P \geqslant P4_j, \quad \text{where } P4_j = \left(2K(S+F_j)/\left(H_2N_j^2\right)\right)^{1/e} \quad \text{from } Q_{2,j}(P) \leqslant N_j, \tag{15}$$

$$P < P5_j, \quad \text{where } P5_j = \left(2K(S+F_j)/\left(H_2N_{j-1}^2\right)\right)^{1/e} \quad \text{from } Q_{2,j}(P) > N_{j-1}, \tag{16}$$

$$P \geqslant P6_j, \quad \text{where } P6_j = \left(Kt/N_j\right)^{1/e} \quad \text{from } N_j \geqslant Dt. \tag{17}$$

We conclude that $Q_{1,j}(P)$ determined with P value which satisfies all the three inequalities (10), (12) and (13) can be a candidate of $Q^*(P)$. In other words, $Q_{1,j}(P)$ can be a candidate of $Q^*(P)$ only if P is an element of price interval $\text{PIQ}_j = \{P \mid P3_j \leqslant P < P2_j \text{ and } P \geqslant P1_j\}$. Utilizing the price ranges in Eqs. (10)–(17), we find the following price intervals which correspond to conditions (C-1) and (C-2).

(PI-1). Price interval on which $Q_{i,j}(P)$ becomes a candidate for $Q^*(P)$:

$$\text{PIQ}_j = \left\{P \mid P3_j \leqslant P < P2_j \text{ and } P \geqslant P1_j\right\} \quad \text{for Case 1}, \tag{18}$$

$$\text{PIQ}_j = \left\{P \mid P4_j \leqslant P < P5_j \text{ and } P < P1_j\right\} \quad \text{for Case 2}. \tag{19}$$

(PI-2). Price interval on which N_j becomes a candidate for $Q^*(P)$:

$$\text{PIN}_j = \left\{P \mid P < P3_j \text{ and } P \geqslant P6_j\right\} \quad \text{for Case 1}, \tag{20}$$

$$\text{PIN}_j = \left\{P \mid P < P4_j \text{ and } P < P6_j\right\} \quad \text{for Case 2}. \tag{21}$$

The price intervals we present have a significant role in solving Model 2. We consider (PI-1) for example. If $P \in \text{PIQ}_j$, $Q_{i,j}(P)$ satisfies condition (C-1) and becomes a candidate for $Q^*(P)$. Substituting Q with $Q_{i,j}(P)$ in $\Pi_{i,j}(P, Q)$, we have a problem of maximizing $\Pi_{i,j}(P, Q_{i,j}(P))$, which is a single variable function. Let $\Pi_{i,j}^0(P) = \Pi_{i,j}(P, Q_{i,j}(P))$, $i = 1, 2$; $j = 1, 2, \ldots, n$. Note that $\Pi_{i,j}^0(P)$ is valid only on the interval $P \in \text{PIQ}_j$. Similarly, if $P \in \text{PIN}_j$, then N_j satisfies condition (C-2). Substituting Q with N_j in $\Pi_{i,j}(P, Q)$, we have $\Pi_{i,j}(P, N_j)$, $i = 1, 2$; $j = 1, 2, \ldots, n$, which is also a single variable function since N_j is a constant. Altogether we have at most $4n$ single variable functions in the form of $\Pi_{i,j}^0(P)$ and $\Pi_{i,j}(P, N_j)$. An optimal solution (P^*, Q^*) which maximizes $\Pi(P, Q)$ is found by searching over $\Pi_{i,j}^0(P)$ and $\Pi_{i,j}(P, N_j)$, and

$$\max_{P,Q} \Pi(P, Q) = \max\left[\max_{\substack{P \in \text{PIQ}_j \\ j}} \Pi_{1,j}^0(P),\ \max_{\substack{P \in \text{PIN}_j \\ j}} \Pi_{1,j}(P, N_j),\ \max_{\substack{P \in \text{PIQ}_j \\ j}} \Pi_{2,j}^0(P),\ \max_{\substack{P \in \text{PIN}_j \\ j}} \Pi_{2,j}(P, N_j)\right]. \tag{22}$$

Now, we are going to investigate the characteristics of $\Pi_{i,j}^0(P)$ and $\Pi_{i,j}(P, N_j)$. With $Q = Q_{i,j}(P)$ as a function of P, the following single variable functions are obtained:

$$\Pi_{1,j}^0(P) = D\{P - C(1 - Rt)\} - \sqrt{2H_1D(S_1 + F_j)}, \tag{23}$$

$$\Pi_{2,j}^0(P) = D\{P - C(1 - It)\} - \sqrt{2H_2D(S + F_j)}, \tag{24}$$

where $j = 1, 2, \ldots, n$ and $D = KP^{-e}$. Utilizing *Mathematica* by Wolfram [12] to obtain derivatives of $\Pi_{i,j}^0(P)$, we have

$$\Pi_{1,j}^0(P)' = D + D'\{P - C(1-Rt)\} - D'(2S_1 - S + F_j)\sqrt{\tfrac{1}{2}H_1/\big(D(S_1+F_j)\big)}\,, \tag{25}$$

$$\Pi_{2,j}^0(P)' = D + D'\{P - C(1-It)\} - D'\sqrt{\tfrac{1}{2}H_2(S+F_j)/D}\,. \tag{26}$$

Also, the second order condition for concavity is

$$\Pi_{1,j}^0(P)'' = eDP^{-2}\{(e-1)P - (e+1)C(1-Rt) - f_1(P)\} < 0, \tag{27}$$

$$\Pi_{2,j}^0(P)'' = eDP^{-2}\{(e-1)P - (e+1)C(1-It) - f_2(P)\} < 0, \tag{28}$$

where

$$f_1(P) = \frac{\sqrt{H_1}\left[(e+2)(S+F_j)^2 + (e+1)DCt^2(R-I)\{3(S+F_j) + DCt^2(R-I)\}\right]}{\sqrt{D\{2(S_1+F_j)\}^3}},$$

$$f_2(P) = \sqrt{\frac{H_2(e+2)^2(S+F_j)}{8D}}\,.$$

For $e \leqslant 1$, $\Pi_{i,j}^0(P)' > 0$ and $\Pi_{i,j}^o(P)$ is an increasing function of P. Thus an optimal value $P_{i,j}$ of $\Pi_{i,j}^0(P)$ occurs at the maximum point of the price interval (PIQ_j) corresponding to $Q_{i,j}(P)$. For $e > 1$, it can be shown that both $f_1(P)$ and $f_2(P)$ become positive and the second order condition is satisfied if $P < C(1-Rt)(e+1)/(e-1)$. Note that, given the second order assumption, $P_{i,j}$ is the one which has the minimum absolute value of $\Pi_{i,j}^0(P)'$ on the price interval (PIQ_j).

Now, with $Q = N_j$, the following results are obtained for $\Pi_{i,j}(P, Q)$:

$$\Pi_{1,j}(P, N_j)' = \frac{D}{P}\left\{(1-e)P + eC(1-Rt) + \frac{e(2S_1 - S + F_j)}{N_j}\right\}, \tag{29}$$

$$\Pi_{1,j}(P, N_j)'' = \frac{eD}{P^2}\left[(e-1)P - (e+1)\left\{C(1-Rt) + \frac{S+F_j}{N_j}\right\} - f_3(P)\right], \tag{30}$$

where $f_3(P) = (2e+1)Ct^2(R-I)D/N_j > 0$,

$$\Pi_{2,j}(P, N_j)' = \frac{D}{P}\left\{(1-e)P + eC(1-It) + \frac{e(S+F_j)}{N_j}\right\}, \tag{31}$$

$$\Pi_{2,j}(P, N_j)'' = \frac{eD}{P^2}\left[(e-1)P - (e+1)\left\{C(1-It) + \frac{S+F_j}{N_j}\right\}\right]. \tag{32}$$

For $e \leqslant 1$, $\Pi_{i,j}(P, N_j)$ is increasing in P and for $e > 1$, $f_3(P) > 0$. It can be shown that if $P < C(1-Rt)(e+1)/(e-1)$ holds, $\Pi_{i,j}(P, N_j)$ is concave. Based on these characteristics of $\Pi_{i,j}(P, N_j)$, an optimal value $P_{i,j}$ of $\Pi_{i,j}(P, N_j)$ can be easily determined on the corresponding price interval PIN_j, $j = 1\ 2, \ldots, n$.

Note that for problem with $e > 1$, the algorithm is valid only when $P < C(1-Rt)(e+1)/(e-1)$.

Now, we present the solution procedure for Model 2.

Solution algorithm for Model 2

Step 1. This step identifies all the candidate values Q_0 of Q satisfying $Q_0 \geqslant \mathrm{Dt}$ (for Case 1). For each Q_0, its optimal value $P_{1,j}$ is determined from the corresponding price interval.

Step 1.1. Determine $P_{1,j}$ which maximizes $\Pi^0_{1,j}(P)$ among the following price intervals: $P \in \mathrm{PIQ}_j$ and $P \leqslant P_u$ with $Q_0 = Q_{1,j}(P)$, $j = 1, 2, \ldots, n$, where P_u is a given upper limit of retail price.

Step 1.2. Determine $P_{1,j}$ which maximizes $\Pi_{1,j}(P, N_j)$ among the following price intervals: $P \in \mathrm{PIN}_j$ and $P \leqslant P_u$ with $Q_0 = N_j$, $j = 1, 2, \ldots, n$.

Step 2. This step identifies all the candidate values Q_0 of Q satisfying $Q_0 < Dt$ (for Case 2). For each Q_0, its optimal value $P_{2,j}$ is determined from the corresponding price interval.

Step 2.1. Determine $P_{2,j}$ which maximizes $\Pi^0_{2,j}(P)$ among the following price intervals: $P \in \mathrm{PIQ}_j$ and $P \leqslant P_u$ with $Q_0 = Q_{2,j}(P)$, $j = 1, 2, \ldots, n$.

Step 2.2. Determine $P_{2,j}$ which maximizes $\Pi_{2,j}(P, N_j)$ among the following price intervals: $P \in \mathrm{PIN}_j$ and $P \leqslant P_u$ with $Q_0 = N_j$, $j = 1, 2, \ldots, n$.

Step 3. Select the optimal retail price (P^*) and lot size (Q^*) which gives the maximum annual net profit among those obtained in the previous steps.

Note that in this algorithm the number of calculations needed to find an optimal solution is $4n$, i.e., $(2n) \times$ number of calculations each for Step 1 and Step 2.

4. Numerical example and sensitivity analysis

To illustrate the solution algorithms, the following problem is considered.
$S = \$50$, $K = 2.5 * 10^5$, $C = \$3$, $H = \$0.1$, $R = 0.15$ $(= 15\%)$, $I = 0.1$ $(= 10\%)$, $t = 0.3$, $N_j = j * 500$, $j = 1, 2, \ldots, 10$, and $F_j = 10 * j * (1 - 0.02 * (j - 1))$, $j = 1, 2, \ldots, 10$.

4.1. Solution with Model 1

The optimal solution with $e = 2.5$ and $P^0 = 5.7$ can be obtained through the following steps:

Step 1.

Step 1.1. Since $Q_{1,0} = 917 \in (N_1, N_2]$, $b = 1$.

Step 1.2. Since $Dt = 967 \in (N_1, N_2]$, $a = 2$ and let $c = \max[2, 1] = 2$.

Step 1.3. Since $Q_{1,2} = 1035 > N_2(= 1000)$ and $Q_{1,3} = 1086 \leqslant N_3(= 1500)$, $k = 3$.

Step 1.4. Since k $(= 3) < n(= 10)$, go to Step 1.5.

Step 1.5. Since $Q_{1,k} \geqslant Dt$, compute the annual net profit with Eq. (1) for $Q_0 = N_2$, $Q_{1,3}$.

Step 2.

Step 2.1. Since $Q_{2,0} = 898 \in (N_1, N_2]$, $b = 1$.

Step 2.2. Since $N_1 = 500 < Dt(= 967)$, $Q_{2,2} = 1059 > N_2(= 1000)$ and $Q_{2,3} = 1127 \leqslant N_3(= 1500)$, $k = 3$.

Step 2.3. Since $Q_{2,3} > Dt$, $Q_0 = N_1$ and go to Step 3.

Step 3. Since $\Pi_{1,2}(P^0, 1000) = 8568 = \max[\Pi_{1,2}(P^0, 1000), \Pi_{1,3}(P^0, 1085), \Pi_{2,1}(P^0, 500)]$, an optimal lot size becomes 1000 with its maximum annual net profit \$8568.

4.2. Solution with Model 2 ($e > 1$)

The solution procedure with $e = 2.5$ and $P_u = C(1 - Rt)(e + 1)/(e - 1) = 6.68$ generates an optimal solution (P^*, Q^*) through the following steps:

Step 1.

Step. 1.1. Solving Eq. (25) numerically in the price interval corresponding to $Q_0 = Q_{1,j}(P)$, $j = 1, 2, \ldots, 10$, we obtain $P_{1,j}$ and these results are presented in Table 1.

Table 1
Results of Step 1

j	$Q = Q_{1,j}(P)$			$Q = N_j$		
	$P \in \mathrm{PIQ}_j$ and $P \leqslant P_u$	$P_{1,j}$	$Q_{1,j}(P_{1,j})$	$P \in \mathrm{PIN}_j$ and $P \leqslant P_u$	$P_{1,j}$	N_j
1	$\emptyset$	–	–	$\emptyset$	–	–
2	[5.83, 6.68]	5.83	999	[5.63, 5.82]	5.63	1000
3	[5.05, 6.01]	5.05 [a]	1313 [a]	$\emptyset$	–	–
4	$\emptyset$	–	–	$\emptyset$	–	–
5	$\emptyset$	–	–	$\emptyset$	–	–
6	$\emptyset$	–	–	$\emptyset$	–	–
7	$\emptyset$	–	–	$\emptyset$	–	–
8	$\emptyset$	–	–	$\emptyset$	–	–
9	$\emptyset$	–	–	$\emptyset$	–	–
10	$\emptyset$	–	–	$\emptyset$	–	–

[a] Optimal solution for Case 1 (Annual net profit = \$8811).

Step 1.2 Solving Eq. (29) numerically in the price interval corresponding to $Q_0 = N_j$, $j = 1, 2, \ldots, 10$, we obtain $P_{1,j}$ and these results are presented in Table 1.

Step 2.

Step 2.1. Solving Eq. (26) numerically in the price interval corresponding to $Q_0 = Q_{2,j}(P)$, $j = 1, 2, \ldots, 10$, we obtain $P_{2,j}$ and these results are presented in Table 2.

Step 2.2. Solving Eq. (31) numerically in the price interval corresponding to $Q_0 = N_j$, $j = 1, 2, \ldots, 10$, we obtain $P_{2,j}$ and these results are presented in Table 2.

Step 3. From the results in Steps 1 and 2, an optimal solution (P^*, Q^*) becomes (4.97, 1000) with its maximum annual net profit \$8836.

4.3. Solution with Model 2 ($e \leqslant 1$)

We solve the problem with $P_u = 300$ and $e = 0.5$. Table 3 shows the results obtained at the end of Step 2. Note that at the end of Step 1, we have $\{P \mid P \in \mathrm{PIQ}_j$ and $P \leqslant P_u\} = \emptyset$ and $\{P \mid P \in \mathrm{PIN}_j$ and

Table 2
Results of Step 2

j	$Q = Q_{2,j}(P)$			$Q = N_j$		
	$P \in \mathrm{PIQ}_j$ and $P \leqslant P_u$	$P_{2,j}$	$Q_{2,j}(P_{2,j})$	$P \in \mathrm{PIN}_j$ and $P \leqslant P_u$	$P_{2,j}$	N_j
1	$\emptyset$	–	–	[0.0, 6.68]	5.23	500
2	$\emptyset$	–	–	[0.0, 5.62]	4.97 [a]	1000 [a]
3	[4.54, 5.04]	4.96	1342	[0.0, 4.53]	4.53	1500
4	[3.76, 4.73]	4.53	1582	[0.0, 3.75]	3.75	2000
5	[3.27, 3.89]	3.76	2094	[0.0, 3.26]	3.26	2500
6	[2.91, 3.36]	3.26	2602	[0.0, 2.90]	2.90	3000
7	[2.65, 2.99]	2.91	3108	[0.0, 2.64]	2.64	3500
8	[2.44, 2.71]	2.65	3611	[0.0, 2.43]	2.43	4000
9	[2.27, 2.49]	2.44	4113	[0.0, 2.26]	2.26	4500
10	[2.13, 2.31]	2.27	4613	[0.0, 2.12]	2.12	5000

[a] Optimal solution for Case 2. This solution is also the global optimum with $\Pi_{2,2}(P_{2,2}, N_2) = \$8836$.

Table 3
Results of the case problem with $P_u = 300$ and $e = 0.5$

j	$Q = Q_{2,j}(P)$				$Q = N_j$			
	$P \in \mathrm{PIQ}_j$ and $P \leqslant P_u$	$P_{2,j}$	$Q_{2,j}(P_{2,j})$	$\Pi^0_{2,j}(P_{2,j})$	$P \in \mathrm{PIN}_j$ and $P \leqslant P_u$	$P_{2,j}$	N_j	$\Pi_{2,j}(P_{2j}, N_j)$
1	$\emptyset$	–	–	–	[0.0, 300.0]	300.0	500	4 286 293
2	$\emptyset$	–	–	–	[0.0, 300.0]	300.0	1000	4 286 920
3	$\emptyset$	–	–	–	[0.0, 300.0]	300.0	1500	4 287 066
4	$\emptyset$	–	–	–	[0.0, 300.0]	300.0 [a]	2000 [a]	4 287 092
5	$\emptyset$	–	–	–	[0.0, 300.0]	300.0	2500	4 287 070
6	[208.6, 300.0]	300.0	2740	4 287 029	[0.0, 208.6]	208.6	3000	4 287 024
7	[129.6, 240.2]	240.2	3001	3 826 456	[0.0, 129.6]	129.6	3500	4 286 964
8	[86.1, 146.9]	146.9	3501	2 968 635	[0.0, 86.1]	86.1	4000	4 286 896
9	[60.1, 96.2]	96.2	4001	2 376 267	[0.0, 60.1]	60.1	4500	4 286 821
10	[43.5, 66.3]	66.3	4501	1 944 474	[0.0, 43.5]	43.5	5000	4 286 737

[a] Optimal solution for Case 2 which also becomes the global optimum.

$P \leqslant P_u\} = \emptyset$, $j = 1, 2, \ldots, n$, for Case 1. An optimal solution (P^*, Q^*) becomes (300, 2000) with its maximum annual net profit \$4 287 092.

4.4. Sensitivity analysis

An interesting question is how much effect the length of credit period has on the retail price, the lot size and the retailer's profit. Since the problem structure of Eqs. (1) and (2) does not permit sensitivity analysis, the same example problem is solved to answer the above question. Six levels of t are adopted, $t = 0$, 0.05, 0.1, 0.15, 0.2, and 0.3. For each level of t, six levels of e, ranging from 0.5 to 3.0 with an increment of 0.5, are tested. For the problem with $e \leqslant 1$, $P_u = 15$, 30, 300 are examined. The results are shown in Table 4 and the following observations can be made which are consistent with our expectations:

(i) With $e > 1$, as either t or e increases, P^* decreases.
(ii) With $e > 1$, as t increases, P^* decreases while $\Pi(P^*, Q^*)$ increases.
(iii) With $e \leqslant 1$, P^* is identical with P_u.
(iv) With e fixed, as t increases, Q^* is nondecreasing.

5. Conclusion

We have analyzed the joint pricing and lot sizing policy of a retailer in an environment in which the retail demand of the product is a constant price elasticity function of the retail price, the freight cost has a quantity discount and the supplier provides a certain fixed credit period for settling the amount the retailer owes to him.

For a retailer who benefits from the supplier's offer of permissible delay in payments, it is not uncommon that he lowers the retail price to a certain degree expecting that he can make more profit by stimulating the customer demand. The ordering cost sometimes depends upon the ordering quantity, owing to discounts allowed by a shipping company for large order. In this regard, we think that the model presented in this paper may be more realistic for some real world problems. Sensitivity analysis with an example problem generated results which are consistent with our expectations.

Table 4
Sensitivity analysis with various values of t and e

e	P_u		t					
			0	0.05	0.1	0.15	0.2	0.3
0.5	15	Q^*	4000	4000	4500	4500	4500	4500
		P^*	15.0	15.0	15.0	15.0	15.0	15.0
		$\Pi(P^*, Q^*)$	771 580	772 837	773 832	774 800	775 768	777 705
	30	Q^*	3000	3500	3500	3500	3500	3500
		P^*	30.0	30.0	30.0	30.0	30.0	30.0
		$\Pi(P^*, Q^*)$	1 229 969	1 230 873	1 231 590	1 232 275	1 232 959	1 234 328
	300	Q^*	1500	1500	2000	2000	2000	2000
		P^*	300.0	300.0	300.0	300.0	300.0	300.0
		$\Pi(P^*, Q^*)$	4 285 655	4 285 953	4 286 215	4 286 443	4 286 659	4 287 092
1.0	15	Q^*	2000	2000	2000	2000	2000	2000
		P^*	15.0	15.0	15.0	15.0	15.0	15.0
		$\Pi(P^*, Q^*)$	198 720	199 069	199 366	199 620	199 870	200 370
	30	Q^*	1500	1500	1500	1500	1500	1500
		P^*	30.0	30.0	30.0	30.0	30.0	30.0
		$\Pi(P^*, Q^*)$	224 150	224 329	224 490	224 634	224 762	225 012
	300	Q^*	426	427	429	431	435	446
		P^*	300.0	300.0	300.0	300.0	300.0	300.0
		$\Pi(P^*, Q^*)$	247 266	247 284	247 302	247 319	247 336	247 367
1.5		Q^*	1500	1500	1500	1500	1500	1500
		P^*	9.16	9.10	9.05	9.02	8.99	8.88
		$\Pi(P^*, Q^*)$	54 663	54 857	55 031	55 185	55 325	55 606
2.0		Q^*	1000	1000	1000	1500	1500	1500
		P^*	6.14	6.10	6.06	6.00	5.97	5.92
		$\Pi(P^*, Q^*)$	20 086	20 228	20 354	20 471	20 585	20 797
2.5		Q^*	1000	1000	1000	1000	1000	1000
		P^*	5.11	5.08	5.05	5.02	5.00	4.97
		$\Pi(P^*, Q^*)$	8367	8459	8546	8627	8701	8836
3.0		Q^*	500	808	825	851	877	950
		P^*	4.68	4.60	4.57	4.53	4.51	4.47
		$\Pi(P^*, Q^*)$	3667	3722	3776	3828	3878	3971

Appendix

Proof of (i) in Theorem 1. Since $Q_{1,k} < Dt$ and $Q_{1,0} > N_b$, the index k equals a. And $Q_{1,a}$ is a maximum point of $\Pi_{1,a}(P^0, Q)$, which is a concave function. Thus, $\Pi_{1,a}(P^0, Q)$ is decreasing in Q for $Q \geqslant Dt$ and by Property 2, we have

$$\Pi_{1,a}(P^0, Dt) \geqslant \Pi_{1,a}(P^0, Q) \quad \text{for } Q \geqslant Dt \tag{A.1}$$

$$> \Pi_{1,\nu}(P^0, Q) \quad \text{for all } \nu > a. \tag{A.2}$$

It can be concluded that Dt gives a maximum annual net profit for $Q \geqslant Dt$.

Also, from Eq. (3),

$$Q_{1,a} = \sqrt{2D(S+F_a) + D^2C(R-I)t^2/[H+CR]} < Dt. \tag{A.3}$$

Squaring both sides of Eq. (A.3) and rearranging,

$$\sqrt{2D(S+F_a)/(H+CI)} < Dt. \tag{A.4}$$

Eq. (A.4) implies that $Q_{2,a} = \sqrt{2D(S+F_a)/(H+CI)} < Dt$ and so $\Pi_{2,a}(P^0, Q)$ is decreasing in Q for $Q_{2,a} < Q < Dt$. Also, since the annual net profit function is continuous at $Q = D_t$, we have

$$\Pi_{1,a}(P^0, Dt) = \Pi_{2,a}(P^0, Dt) < \Pi_{2,a}(P^0, Q) \quad \text{for some } Q < Dt. \tag{A.5}$$

Therefore, if $Q_{1,a} < Dt$, then Q^* must be less than Dt. □

Proof of (i) in Theorem 2. Since $Q_{2,0} < Q_{2,j}$ for $j = 1, 2, \ldots, n$, we have

$$Q_{2,\nu} > N_\nu \quad \text{for any } \nu \leqslant b. \tag{A.6}$$

Note that $Q_{2,\nu}$ is a maximum point of $\Pi_{2,\nu}(P^0, Q)$, which is a concave function. Thus,

$$\Pi_{2,\nu}(P^0, N_\nu) \geqslant \Pi_{2,\nu}(P^0, Q) \quad \text{for } Q \in (N_{\nu-1}, N_\nu). \tag{A.7}$$

Also, from Eq. (2),

$$\Pi_{2,\nu-1}(P^0, N_{\nu-1}) = D(P-C) - \frac{N_{\nu-1}H}{2} - \frac{DS}{N_{\nu-1}} - \left(\frac{N_{\nu-1}IC}{2} - DCIt\right) - \frac{DF_{\nu-1}}{N_{\nu-1}} \tag{A.8}$$

$$< D(P-C) - \frac{N_\nu H}{2} - \frac{DS}{N_\nu} - \left(\frac{N_\nu IC}{2} - DCIt\right) - \frac{DF_{\nu-1}}{N_{\nu-1}}$$

$$(\text{because } N_\nu \leqslant N_b < Q_{1,0}) \tag{A.9}$$

$$< D(P-C) - \frac{N_\nu H}{2} - \frac{DS}{N_\nu} - \left(\frac{N_\nu IC}{2} - DCIt\right) - \frac{DF_\nu}{N_\nu} = \Pi_{2,\nu}(P^0, N_\nu) \tag{A.10}$$

$$(\text{because } F_{\nu-1}/N_{\nu-1} > F_\nu/N_\nu).$$

From (A.8), (A.9) and (A.10),

$$\Pi_{2,\nu-1}(P^0, N_{\nu-1}) < \Pi_{2,\nu}(P^0, N_\nu) \quad \text{for any } \nu \leqslant b. \tag{A.11}$$

Also, since the annual net profit function is continuous at $Q = Dt$, we have

$$\Pi_{1,a}(P^0, Dt) = \Pi_{2,a}(P^0, Dt) > \Pi_{2,a}(P^0, Q) \quad \text{for } N_{a-1} < Q < Dt. \tag{A.12}$$

Therefore, if $N_b \geqslant Dt$ and $Q^* < Dt$, then $Q^* = N_{a-1}$. □

Acknowledgments

The authors would like to thank the anonymous reviewers for their helpful and constructive comments on the original draft of this paper.

References

[1] Abad, P.L., "Determining optimal selling price and lot size when the supplier offers all-unit quantity discounts", *Decision Sciences* 19 (1988) 622–634.

[2] Abad, P.L., "Joint price and lot-size determination when supplier offers incremental quantity discounts", *Journal of the Operational Research Society* 39 (1988) 603–607.

[3] Das, C., "A unified approach to the price-break economic order quantity (EOQ) problem", *Decision Sciences* 15 (1984) 350–358.

[4] Goyal, S.K., "Economic order quantity under conditions of permissible delay in payments", *Journal of the Operational Research Society* 36 (1985) 335–338.

[5] Hadley, G., and Within, T.M., *Analysis of Inventory Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1963.
[6] Haley, C.W., and Higgins, R.C., "Inventory policy and trade credit financing", *Management Science* 20 (1973) 464–471.
[7] Johnson, L.A., and Montgomery, D.C., *Operations Research in Production, Planning, Scheduling and Inventory Control*, Wiley, New York, 1974.
[8] Kunreuther, H., and Richard, J.F., "Optimal pricing and inventory decisions for non-seasonal items", *Econometrica* 39 (1971) 173–175.
[9] Lee, C.Y., "The economic order quantity for freight discount costs", *IIE Transactions* 18 (1986) 318–320.
[10] Mehta, D., "The formulation of credit policy models", *Management Science* 15 (1968) B30–B50.
[11] Tersine, R.J., and Toelle, R.A., "Lot size determination with quantity discounts", *Production and Inventory Management* 26 (1985) 1–23.
[12] Wolfram, S., *Mathematica*, Addison-Wesley, Reading, MA, 1988.