IE531 Linear Programming HW3

(due : 2018/04/09, Mon.), in class

1. Use a theorem of alternatives to demonstrate that (1, 1, 0) $\notin S(A)$ where $S(A)$ is the subspace generated by rows of A and $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -3 & 1 \end{bmatrix}$.

2. Prove the lemma : Let S be a nonempty subspace of $\mathcal{R}^n$. Then for any $x \in \mathcal{R}^n$, we can decompose x uniquely as $x = x' + x''$, where $x' \in S$ and $x'' \in S^0$.
(Hint: Use number 3 of previous HW.)

3. Consider the subspace alternative theorem. For $A : m \times n$ and $c \in \mathcal{R}^n$, exactly one of the followings holds:
   (I) $\exists \quad y \in \mathcal{R}^m$ s.t. $y'A = c$
   (II) $\exists \quad x \in \mathcal{R}^n$ s.t. $Ax = 0, c'x \neq 0$
   Develop a proof for this along the following lines:
   First observe that clearly both (I) and (II) cannot hold simultaneously, so that it suffices to show that when (I) fails, (II) must hold. Next assume (I) fails and use the lemma of #2 to determine an x which satisfies (II).

4. Let $a \in L$, where L is an affine space in $\mathcal{R}^n$. Show that $L = S + a \equiv \{x + a : x \in S\}$ for some subspace $S \subseteq \mathcal{R}^n$. Also show that S is uniquely determined by L;, i.e., independent of the choice of $a \in L$.

5. For $A : m \times n, b \in \mathcal{R}^m$, show that there is a unique solution to the system $Ax = b$ if and only if there is no nonzero solution to $Ax = 0$. (Assume $Ax = b$ has a solution.)

6. Let $a^1, \ldots, a^m \in \mathcal{R}^n$. Show that
   (a) $a^1, \ldots, a^m$ are affinely independent if and only if $\sum_{i=1}^m \lambda_i a^i = 0, \sum_{i=1}^m \lambda_i = 0$ implies $\lambda_i = 0$ for $1 \leq i \leq m$.
   (b) $a^1, \ldots, a^m$ are affinely independent if and only if for each k, the vectors $a^i - a^k$ for $i \neq k$ are linearly independent.
   (c) $a^1, \ldots, a^m$ are affinely independent if and only if $(a^1, -1), \ldots, (a^m, -1) \in \mathcal{R}^{n+1}$ are linearly independent.

7. Show that if $A : m \times n$ with $m < n$, then the homogeneous system $Ax = 0$ always has a solution $x \neq 0$.