IE 531

Linear Programming HW2

(due : 2018/04/02, Mon), in class

1. Show that the closure axiom for vector spaces, i.e. for $\mathcal{V}$ a vector space, $a^1, a^2 \in \mathcal{V}, \lambda \in \mathcal{R} \Rightarrow a^1 + \lambda a^2 \in \mathcal{V}$ is equivalent to $\lambda_1 a^1 + \cdots + \lambda_m a^m \in \mathcal{V}$ for all $m \geq 1$, for all $\lambda_1, \cdots, \lambda_m \in \mathcal{R}$ and for all $a^1, \cdots, a^m \in \mathcal{V}$.

2. For any $A \subseteq \mathcal{R}^n$, show that the linear span and linear hull of A are subspaces, in fact the same subspaces of $\mathcal{R}^n$.

3. Let $A \subseteq \mathcal{R}^n$. Show :
   (a) A^o is a subspace
   (b) $A \cap A^o \subseteq \{0\}$
   (c) If $\{a^1, \cdots, a^m\}$ are linearly independent in A and $\{a^{m+1}, \cdots, a^p\}$ are linearly independent in A^o, then $\{a^1, \cdots, a^p\}$ are linearly independent in $\mathcal{R}^n$.
   (d) If $B \subseteq A$ and B generates A, then $A^o = B^o$.

4. Show that the following elementary row operations on matrix A leave $\mathcal{S}(A)$ unchanged. ($\mathcal{S}(A)$ is the subspace generated by the row vectors of A. Here a_p means the pth row of A.)
   (a) Interchange rows a_i and a_k.
   (b) Replace row a_i by λa_i where $\lambda \neq 0$ and $\lambda \in \mathcal{R}$.
   (c) Replace row a_k by $a_k + \lambda a_i$ where $\lambda \in \mathcal{R}$.

5. Prove : Let $a^1, \cdots, a^m \in \mathcal{R}^n$ are lin. indep and $a^0 = \sum_{i=1}^m \lambda_i a^i$.
   Then (1) all λ_i unique and (2) $\{a^1, \cdots, a^m\} \cup \{a^0\} \backslash \{a^k\}$ lin. indep $\Leftrightarrow \lambda_k \neq 0$.

6. Let $A : m \times n$ be given. Suppose after performing Gauss-Jordan Elimination, the resulting matrix is EA. Let B be the $m \times m$ matrix whose kth column is given by column j_k of A. Thus for some permutation matrix P, we can partition $AP = \begin{bmatrix} B : N \end{bmatrix}$. Through Gauss-Jordan Elimination, we also have $EAP = \begin{bmatrix} I_m : EN \end{bmatrix}$; thus implying $E = B^{-1}$.

Note : The set of row vectors of the matrix $B^{-1}A$ is also a basis for $\mathcal{S}(A)$ and is called the standard representation matrix for $\mathcal{S}(A)$ with respect to column basis B.

Consider matrix $D \equiv P \left[\begin{array}{c} -B^{-1}N \\ \hline I_{n-m} \end{array} \right]$.

(a) Show that the columns of D are in A^o.

(b) Conclude from (a) above and (3-c) that the columns of D constitute a basis for A^o. (Note : You have just(basically) proved rank(A) + nullity$(A) = n$)