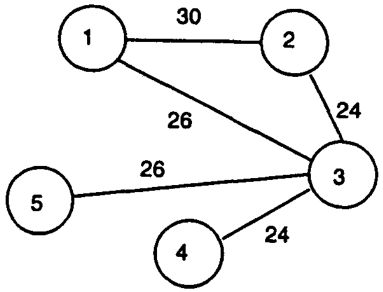

Fig. 10.3 Optimal 1-tree for $\bar{c}^1$

Iteration 2. $u^2 = (0,0,-6,3,3)$. The new cost matrix is

$$(\bar{c}^2_e) = \begin{pmatrix} - & 30 & 32 & 47 & 37 \\ - & - & 30 & 37 & 47 \\ - & - & - & 27 & 29 \\ - & - & - & - & 24 \\ - & - & - & - & - \end{pmatrix}.$$

We obtain $z(u^2) = 143 + 2\sum_i u^2_i = 143$, and

$$u^3 = u^2 + ((148 - 143)/2)(0,0,-1,0,1).$$

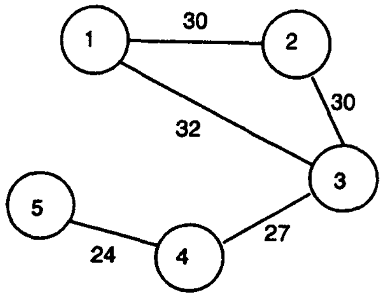

Fig. 10.4 Optimal 1-tree for $\bar{c}^2$

The new optimal 1-tree is shown in Figure 10.4.

Iteration 3. $u^3 = (0,0,-17/2,3,11/2)$.
The new cost matrix is

$$(\bar{c}^3_e) = \begin{pmatrix} - & 30 & 34.5 & 47 & 34.5 \\ - & - & 32.5 & 37 & 44.5 \\ - & - & - & 29.5 & 29 \\ - & - & - & - & 21.5 \\ - & - & - & - & - \end{pmatrix}.$$

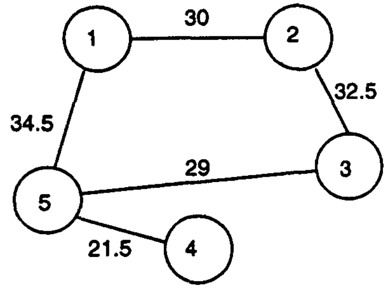

Fig. 10.5 Optimal 1-tree for $\bar{c}^3$

The new optimal 1-tree is shown in Figure 10.5 and we obtain the lower bound $z(u^3) = 147.5$. As the cost data c are integral, we know that z is integer valued and so $z \geq \lceil 147.5 \rceil = 148$. As a solution of cost 148 is known, the corresponding solution has been proved optimal. ∎

As the subgradient algorithm is often terminated before the optimal value w_{LD} is attained, and also as there is in most cases a duality gap ($w_{LD} > z$), Lagrangian relaxation must typically be embedded in a branch-and-bound algorithm.

10.4 LAGRANGIAN HEURISTICS AND VARIABLE FIXING

Once the dual variables u begin to approach the set of optimal solutions, a solution $x(u)$ is obtained that is hopefully "close" to being primal feasible every time that a Lagrangian subproblem $IP(u)$ is solved. In the $STSP$ many nodes of the 1-tree will have degree 2, and so the solution is not far from being a tour, while for the UFL, many clients are served exactly once, and only a few are not served at all. Therefore it is often straightforward to devise a simple heuristic that converts $x(u)$ into a feasible solution without greatly decreasing/increasing its value/cost. Below we examine this simple idea for set covering problems, as well as the possibility of fixing some variables once good primal and dual solutions are available.

Consider an instance of the set-covering problem

$$\min\{\sum_{j \in N} c_j x_j : \sum_{j \in N} a_{ij} x_j \geq 1 \text{ for } i \in M, x \in B^n\},$$

with $a_{ij} \in \{0,1\}$ for $i \in M, j \in N$. The Lagrangian relaxation in which all the covering constraints are dualized is

$$z(u) = \sum_{i \in M} u_i + \min\{\sum_{j \in N}(c_j - \sum_{i \in M} u_i a_{ij})x_j : x \in B^n\}$$

for $u \geq 0$.

One simple possibility is to take an optimal solution $x(u)$ of this relaxation, drop all rows covered by the solution $x(u)$, that is, the rows $i \in M$ for which $\sum_{j\in N} a_{ij}x_j(u) \geq 1$, and solve the remaining smaller covering problem by a greedy heuristic. If y^* is the heuristic solution, then $x^H = x(u) + y^*$ is a feasible solution. It is then worth checking whether it cannot be improved by setting to zero some of the variables with $x_j(u) = 1$.

Once a heuristic solution has been found, it is also possible to use the Lagrangian for variable fixing. If $\overline{z}$ is the incumbent value, then any better feasible solution x satisfies $\sum_{i\in M} u_i + \min \sum_{j\in N}(c_j - \sum_{i\in M} u_i a_{ij})x_j \leq cx < \overline{z}$. Let $N_1 = \{j \in N : c_j - \sum_{i\in M} u_i a_{ij} > 0\}$ and $N_0 = \{j \in N : c_j - \sum_{i\in M} u_i a_{ij} < 0\}$.

Proposition 10.5 *If $k \in N_1$ and $\sum_{i\in M} u_i + \sum_{j\in N_0}(c_j - \sum_{i\in M} u_i a_{ij}) + (c_k - \sum_{i\in M} u_i a_{ik}) \geq \overline{z}$, then $x_k = 0$ in any better feasible solution.*

If $k \in N_0$ and $\sum_{i\in M} u_i + \sum_{j\in N_0\setminus\{k\}}(c_j - \sum_{i\in M} u_i a_{ij}) \geq \overline{z}$, then $x_k = 1$ in any better feasible solution.

Example 10.3 Consider a set-covering instance with $m = 4, n = 6$,

$$c = (6,6,11,5,8,8) \text{ and } a_{ij} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Taking $u = (4,4,3,3)$, the Lagrangian subproblem $IP(u)$ takes the form

$$z(u) = 14 + \min\{-1x_1 + 0x_2 + 1x_3 + 1x_4 + 1x_5 + 1x_6 : x \in B^6\}.$$

An optimal solution is clearly $x(u) = (1,0,0,0,0,0)$ wth $z(u) = 13$. The solution $x(u)$ covers rows 1 and 3, so the remaining problem to be solved heuristically is

$$\begin{array}{lllllllllll} \min 6x_1 + & 6x_2 & + & 11x_3 & + & 5x_4 & + & 8x_5 & + & 8x_6 & \\ & x_2 & & & & x_4 & + & x_5 & & & \geq 1 \\ & & & x_3 & & & & x_5 & & & \geq 1 \\ & & & & x & \in & B^6 & & & & \end{array}$$

for which a greedy heuristic gives the solution $y^* = (0,0,0,0,1,0)$. Adding together these two vectors, we obtain the heuristic solution $x^H = (1,0,0,0,1,0)$ with cost 14. Thus we now know that the optimal value lies between 13 and 14.

Now using Proposition 10.5, we see that with $N_0 = \{1\}$ and $N_1 = \{3,4,5,6\}$, $x_1 = 1$ and $x_3 = x_4 = x_5 = x_6 = 0$ in any solution whose value is less than 14. ∎

10.5 CHOOSING A LAGRANGIAN DUAL

Suppose the problem to be solved is of the form:

$$(IP) \qquad \begin{array}{c} z = \max cx \\ A^1x \leq b^1 \\ A^2x \leq b^2 \\ x \in Z^n_+. \end{array}$$

If one wishes to tackle the problem by Lagrangian relaxation, there is a choice to be made. Should one dualize one or both sets of constraints, and if so, which sets? The answer must be based on a trade-off between

(i) the *strength* of the resulting Lagrangian dual bound w_{LD},

(ii) *ease of solution* of the Lagrangian subproblems $IP(u)$, and

(iii) *ease of solution* of the Lagrangian dual problem: $w_{LD} = \min_{u\geq 0} z(u)$.

Concerning (i), Theorem 10.3 gives us precise information about the strength of the bound.

Concerning (ii), the ease of solution of $IP(u)$ is problem specific. However, we know that if $IP(u)$ is "easy" in the sense of reducing to a linear program, that is $IP(u)$ involves maximization over $X = \{x \in Z^n_+ : Ax \leq b\}$ and $\text{conv}(X) = \{x \in R^n_+ : Ax \leq b\}$, then solving the linear programming relaxation of IP is an alternative to Lagrangian relaxation.

Concerning (iii), the difficulty using the subgradient (or other) algorithms is hard to estimate a priori, but the number of dual variables is at least some measure of the probable difficulty.

To demonstrate these trade-offs, consider the *Generalized Assignment Problem (GAP)*:

$$\begin{array}{c} z = \max \sum_{j=1}^n \sum_{i=1}^m c_{ij}x_{ij} \\ \sum_{j=1}^n x_{ij} \leq 1 \text{ for } i = 1,\ldots,m \\ \sum_{i=1}^m a_{ij}x_{ij} \leq b_j \text{ for } j = 1,\ldots,n \\ x \in B^{mn}. \end{array}$$

We consider three possible Lagrangian relaxations. In the first we dualize both sets of constraints giving $w^1_{LD} = \min_{u\geq 0, v\geq 0} w^1(u,v)$ where

$$\begin{array}{c} w^1(u,v) = \max_x \sum_{j=1}^n \sum_{i=1}^m (c_{ij} - u_i - a_{ij}v_j)x_{ij} + \sum_{i=1}^m u_i + \sum_{j=1}^n v_j b_j \\ x \in B^{mn}. \end{array}$$

Here we dualize the first set of assignment constraints giving $w^2_{LD} = \min_{u\geq 0} w^2(u)$ where

$$w^2(u) = \max_x \sum_{j=1}^n \sum_{i=1}^m (c_{ij} - u_i)x_{ij} + \sum_{i=1}^m u_i$$
$$\sum_{i=1}^m a_{ij}x_{ij} \leq b_j \text{ for } j = 1, \ldots, n$$
$$x \in B^{mn},$$

and here we dualize the knapsack constraints giving $w^3_{LD} = \min_{v \geq 0} w^3(v)$ where

$$w^3(v) = \max_x \sum_{j=1}^n \sum_{i=1}^m (c_{ij} - a_{ij}v_j)x_{ij} + \sum_{j=1}^n v_j b_j$$
$$\sum_{j=1}^n x_{ij} \leq 1 \text{ for } i = 1, \ldots, m$$
$$x \in B^{mn}.$$

Based on Theorem 10.3, we know that $w^1_{LD} = w^3_{LD} = z_{LP}$ as for each i, $\text{conv}\{x : \sum_{j=1}^n x_{ij} \leq 1, x_{ij} \in \{0,1\} \text{ for } j = 1, \ldots, n\} = \{x : \sum_{j=1}^n x_{ij} \leq 1, 0 \leq x_{ij} \leq 1 \text{ for } j = 1, \ldots, n\}$. The values of $w^1(u,v)$ and $w^3(v)$ can both be calculated by inspection. To calculate $w^1(u,v)$, note that the problem decomposes variable by variable, while for $w^3(u,v)$ the problem decomposes into a simple problem for each $j = 1, \ldots, n$. In terms of solving the Lagrangian dual problems, calculating w^3_{LD} appears easier than calculating w^1_{LD} because there are only m as opposed to $m + n$ dual variables.

The second relaxation potentially gives a tighter bound $w^2_{LD} \leq z_{LP}$ as in general for fixed j, $\text{conv}\{x : \sum_{i=1}^m a_{ij}x_{ij} \leq b_j, x_{ij} \in \{0,1\}^m\} \subset \{x : \sum_{i=1}^m a_{ij}x_{ij} \leq b_j, 0 \leq x_{ij} \leq 1 \text{ for } i = 1, \ldots, m\}$. However, here the Lagrangian subproblem involves the solution of m 0–1 knapsack problems.

10.6 NOTES

10.1 Many of the properties of the Lagrangian dual can be found in [Eve63]. The successful solution of what were at the time very large *TSPs* [HelKar70], [HelKar71] made the approach popular.

10.2 The application to integer programming and in particular Theorem 10.3 and its consequences were explored in [Geo74].

10.3 The use of the subgradient algorithm to solve the Lagrangian dual again stems from [HelKar70]. Detailed studies and analysis of the subgradient approach can be found in [HelWolCro74] and [Gof77]. Recently more sophisticated nondifferentiable optimization techniques have been used; see [Lemetal95]. Simple multiplier adjustment methods have also been tried for various problems, among them the uncapacitated facility location problem [Erl78]. An alternative approach, Dantzig-Wolfe decomposition, is treated in the next chapter.

10.5 A comparison of different Lagrangian relaxations for the capacitated facility location problem can be found in [CorSriThi91]. By duplicating variables, dualizing the equations identifying variables, and solving separate subproblems for each set of distinct variables, Lagrangian relaxation can be used to get stronger bounds in certain cases; see Exercise 10.6. Lagrangian decomposition is one of several names given to this idea [JorNas86], [GuiKim87].

Lagrangian relaxation is an important practical tool for many structured problems. Surveys on the applications of Lagrangian duality include [Fis81] and [Beas93]. It suffices to open journals such as *Operations Research, Management Science* or the *European Journal of Operations Research* to find a wide variety of applications. [Beas93] lists 21 applications based on Lagrangian relaxation that were found in these three journals just for 1991.

10.7 EXERCISES

1. Consider an instance of UFL with $m = 6, n = 5$, delivery costs

$$c_{ij} = \begin{pmatrix} 6 & 2 & 1 & 3 & 5 \\ 4 & 10 & 2 & 6 & 1 \\ 3 & 2 & 4 & 1 & 3 \\ 2 & 0 & 4 & 1 & 4 \\ 1 & 8 & 6 & 2 & 5 \\ 3 & 2 & 4 & 8 & 1 \end{pmatrix}$$

and fixed costs $f = (4, 8, 11, 7, 5)$. Using the dual vector $u = (5, 6, 3, 2, 6, 4)$, solve the Lagrangian subproblem $IP(u)$ to get an optimal solution $(x(u), y(u))$ and lower bound $z(u)$. Modify the dual solution $(x(u), y(u))$ to construct a good primal feasible solution. How far is this solution from optimal?

2. Suppose one dualizes the constraints $x_{ij} \leq y_j$ in the strong formulation of UFL. How strong is the resulting Lagrangian dual bound, and how easy is the solution of the Lagrangian subproblem?

3. Use Lagrangian relaxation to solve the $STSP$ instance with distances

$$(c_e) = \begin{pmatrix} - & 8 & 2 & 14 & 26 & 13 \\ - & - & 7 & 4 & 16 & 8 \\ - & - & - & 23 & 14 & 9 \\ - & - & - & - & 12 & 6 \\ - & - & - & - & - & 5 \end{pmatrix}.$$