The minimal description is given by

$$\begin{array}{rcrcl} x_1 & & & \leq & 2 \\ 2x_1 & + & x_2 & \leq & 6 \\ x_1 & + & x_2 & \geq & 2 \\ x_1 & & & \geq & 0. \end{array}$$

■

9.2.3 Facet and Convex Hull Proofs*

This section is for those interested in proving results about the strength of certain inequalities or formulations. The aim is to indicate ways to show that a valid inequality is facet-defining, or that a set of inequalities describes the convex hull of some discrete set $X \subset Z^n_+$.

For simplicity we assume throughout this subsection that conv(X) is bounded as well as full-dimensional. So there are no hyperplanes containing all the points of X. As example we take the set $X = \{(x, y) \in R^m_+ \times B^1 : \sum_{i=1}^m x_i \leq my\}$ that arises in Sections 1.6 and 8.4 in formulating the uncapacitated facility location problem.

Problem 1. Given $X \subset Z^n_+$ and a valid inequality $\pi x \leq \pi_0$ for X, show that the inequality defines a facet of conv(X).

We consider two different approaches.

Approach 1. (Just use the definition.) Find n points $x^1, \ldots, x^n \in X$ satisfying $\pi x = \pi_0$, and then prove that these n points are affinely independent.

Approach 2. (An indirect but useful way to verify the affine independence.)
(i) Select $t \geq n$ points $x^1, \ldots, x^t \in X$ satisfying $\pi x = \pi_0$. Suppose that all these points lie on a generic hyperplane $\mu x = \mu_0$.
(ii) Solve the linear equation system

$$\sum_{j=1}^{n} \mu_j x_j^k = \mu_0 \text{ for } k = 1, \ldots, t$$

in the $n+1$ unknowns (μ, μ_0).
(iii) If the only solution is $(\mu, \mu_0) = \lambda(\pi, \pi_0)$ for $\lambda \neq 0$, then the inequality $\pi x \leq \pi_0$ is facet-defining.

Example 9.3 Taking $X = \{(x, y) \in R^m_+ \times B^1 : \sum_{i=1}^m x_i \leq my\}$, we have that $\dim(\text{conv}(X)) = m + 1$. Now we consider the valid inequality $x_i \leq y$ and show that it is facet-defining using Approach 2.

We select the simplest points $(\mathbf{0}, 0), (e_i, 1)$ and $(e_i + e_j, 1)$ for $j \neq i$ that are feasible and satisfy $x_i = y$.

As $(0,0)$ lies on $\sum_{i=1}^m \mu_i x_i + \mu_{m+1} y = \mu_0$, $\mu_0 = 0$.
As $(e_i, 1)$ lies on the hyperplane $\sum_{i=1}^m \mu_i x_i + \mu_{m+1} y = 0$, $\mu_i = -\mu_{m+1}$.
As $(e_i + e_j, 1)$ lies on the hyperplane $\sum_{i=1}^m \mu_i x_i - \mu_i y = 0$, $\mu_j = 0$ for $j \neq i$.
So the hyperplane is $\mu_i x_i - \mu_i y = 0$, and $x_i \leq y$ is facet-defining. ∎

Problem 2. Show that the polyhedron $P = \{x \in R^n : Ax \leq b\}$ describes conv(X).

Here we present eight approaches.

Approach 1. Show that the matrix A, or the pair (A, b) have special structure guaranteeing that $P =$conv(X).

Example 9.4 Take $X = \{(x, y) \in R_+^m \times B^1 : \sum_{i=1}^m x_i \leq my\}$, and consider the polyhedron/formulation

$$P = \{(x, y) \in R_+^m \times R^1 : x_i \leq y \text{ for } i = 1, \ldots, m, y \leq 1\}.$$

Observe that the constraints $x_i - y \leq 0$ for $i = 1, \ldots, m$ lead to a matrix with a coefficient of $+1$ and -1 in each row. Such a matrix is TU; see Proposition 3.2. Adding the bound constraints still leaves a TU matrix. Now as the requirements vector is integer, it follows from Proposition 3.3 that all basic solutions are integral, and $P =$ conv(X). ∎

Approach 2. Show that points $(x, y) \in P$ with y fractional are not extreme points of P.

Example 9.4 (cont) Suppose that $(x^*, y^*) \in P$ with $0 < y^* < 1$. Note first that $(\mathbf{0}, 0) \in P$. Also as $x_i^* \leq y^*$, the point $(\frac{x_1^*}{y^*}, \ldots, \frac{x_m^*}{y^*}, 1) \in P$. But now

$$(x^*, y^*) = (1 - y^*)(\mathbf{0}, 0) + y^*(\frac{x_1^*}{y^*}, \ldots, \frac{x_m^*}{y^*}, 1)$$

is a convex combination of two points of P and is not extreme. Thus all vertices of P have y^* integer. ∎

Approach 3. Show that for all $c \in R^n$, the linear program $z^{LP} = \max\{cx : Ax \leq b\}$ has an optimal solution $x^* \in X$.

Example 9.4 (cont) Consider the linear program $z^{LP} = \max\{\sum_{i=1}^m c_i x_i + fy : 0 \leq x_i \leq y \text{ for } i = 1, \ldots, m, y \leq 1\}$. Consider an optimal solution (x^*, y^*). Because of the constraints $0 \leq x_i \leq y$, any optimal solution has $x_i^* = y^*$ if $c_i > 0$ and $x_i^* = 0$ if $c_i < 0$. The corresponding solution value is $(\sum_{i:c_i>0} c_i + f)y^*$ if $y^* > 0$ and 0 otherwise. Obviously if $(\sum_{i:c_i>0} c_i + f) > 0$, the objective is maximized by setting $y^* = 1$, and otherwise $y^* = 0$ is optimal. Thus there is always an optimal solution with y integer, and $z^{LP} = (\sum_{i:c_i>0} c_i + f)^+$. ∎

Approach 4. Show that for all $c \in R^n$, there exists a point $x^* \in X$ and a feasible solution u^* of the dual LP $w^{LP} = \min\{ub, uA = c, u \geq 0\}$ with $cx^* = u^*b$. Note that this implies that the condition of Approach 3 is satisfied.

Example 9.4 (cont) The dual linear program is

$$\begin{aligned} &\min t \\ &w_i \geq c_i \text{ for } i = 1, \dots, m \\ &-\textstyle\sum_{i=1}^m w_i + t \geq f \\ &w_i \geq 0 \text{ for } i = 1, \dots, m, t \geq 0. \end{aligned}$$

Consider the two points $(\mathbf{0}, 0)$ and $(x^*, 1)$ with $x_i^* = 1$ if $c_i > 0$ and $x_i^* = 0$ otherwise. Taking the better of the two leads to a primal solution of value $(\sum_{i:c_i>0} c_i + f)^+$. The point $w_i = c_i^+$ for $i = 1, \dots, m$ and $t = (\sum_{i:c_i>0} c_i + f)^+$ is clearly feasible in the dual. Thus we have found a point in X and a dual solution of the same value. ∎

Approach 5. Show that if $\pi x \leq \pi_0$ defines a facet of conv(X), then it must be identical to one of the inequalities $a^i x \leq b_i$ defining P.

Example 9.4 (cont) Consider the inequality $\sum_{i=1}^m \pi_i x_i + \pi_{m+1} y \leq \pi_0$. Let $S = \{i \in \{1, \dots, m\} : \pi_i > 0\}$ and $T = \{i \in \{1, \dots, m\} : \pi_i < 0\}$. Note that as the point $(\mathbf{0}, 0) \in X$, $\pi_0 \geq 0$, and as $(e^S, 1) \in X$, $\sum_{i \in S} \pi_i + \pi_{m+1} \leq \pi_0$, where e^S is the characteristic vector of S. Also a facet-defining inequality must have a tight point with $y = 1$. The point $(e^S, 1)$ maximizes the lhs, and so $\sum_{i \in S} \pi_i + \pi_{m+1} = \pi_0 \geq 0$.

Now consider the valid inequality obtained as a nonnegative combination of valid inequalities:

$x_i - y \leq 0$ with weight π_i for $i \in S$

$-x_i \leq 0$ with weight $-\pi_i$ for $i \in T$

$y \leq 1$ with weight $\sum_{i \in S} \pi_i + \pi_{m+1}$.

The resulting inequality is $\sum_{i=1}^m \pi_i x_i + \pi_{m+1} y \leq \sum_{i \in S} \pi_i + \pi_{m+1}$. This dominates or equals the original inequality as $\sum_{i \in S} \pi_i + \pi_{m+1} \leq \pi_0$. So the only inequalities that are not nonnegative combinations of other inequalities are those describing P. ∎

Approach 6. Show that for any $c \in R^n, c \neq 0$, the set of optimal solutions $M(c)$ to the problem $\max\{cx : x \in X\}$ lies in $\{x : a^i x = b_i\}$ for some $i = 1, \dots, m$, where $a^i x \leq b_i$ for $i = 1, \cdots, m$ are the inequalities defining P.

Example 9.4 (cont) Consider an arbitrary objective $(c, f) \in R^m \times R^1$.

If $f > 0$, $y = 1$ in every optimal solution and so $M(c, f) \in \{(x, y) : y = 1\}$.

If $c_i < 0$, then $x_i = 0$ in every optimal solution.

If $c_i > 0$ and $f \leq 0$, then $x_i = y$ in every optimal solution.

If $c_i = 0$ for all i and $f < 0$, then $x_i = 0$ in any optimal solution.

All cases have been covered, and so $P = \text{conv}(X)$. ∎

Approach 7. Verify that $b \in Z^n$, and show that for all $c \in Z^n$, the optimal value of the dual w^{LP} is integer valued. This is to show that the inequalities $Ax \leq b$ form a TDI system, see Theorem 3.14.

Example 9.4 (cont) We have shown using Approach 4 that $w^{LP} = (\sum_{i:c_i>0} c_i + f)^+$. This is integer valued when c and f are integral. ∎

Approach 8. (Projection from an Extended Formulation). Suppose $Q \subseteq R^n \times R^p$ is a polyhedron with $P = proj_x(Q)$ as defined in Section 1.7. Show that for all $c \in R^n$, the linear program $\max\{cx : (x, w) \in Q\}$ has an optimal solution with $x \in X$.

Example 9.5 (Uncapacitated Lot-Sizing). It can be shown that solving the extended formulation presented in Section 1.6 as a linear program gives a solution with the set-up variables $y_1, \ldots, y_n$ integral, and thus provides an optimal solution to ULS. So its projection to the (x, y, s) space describes the convex hull of solutions to ULS. ∎

9.3 0–1 KNAPSACK INEQUALITIES

Consider the set $X = \{x \in B^n : \sum_{j=1}^n a_j x_j \leq b\}$. Complementing variables if necessary by setting $\bar{x}_j = 1 - x_j$, we assume throughout this section that the coefficients $\{a_j\}_{j=1}^n$ are positive. Also we assume $b > 0$. Let $N = \{1, \ldots, n\}$.

9.3.1 Cover Inequalities

Definition 9.6 A set $C \subseteq N$ is a *cover* if $\sum_{j \in C} a_j > b$. A cover is *minimal* if $C \setminus \{j\}$ is not a cover for any $j \in C$.

Note that C is a cover if and only if its associated incidence vector x^C is infeasible for S.

Proposition 9.3 *If $C \subseteq N$ is a cover, the cover inequality*

$$\sum_{j \in C} x_j \leq |C| - 1$$

is valid for X.

Proof. We show that if x^R does not satisfy the inequality, then $x^R \notin X$. If $\sum_{j \in C} x_j^R > |C| - 1$, then $|R \cap C| = |C|$ and thus $R \supseteq C$. Then $\sum_{j=1}^n a_j x_j^R = \sum_{j \in R} a_j \geq \sum_{j \in C} a_j > b$ and so $x^R \notin X$. ∎